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"x^2 OR $2x$?": EXPERIMENTS ON GEOMETRIC ALGEBRA WITH DYNAMIC GEOMETRY

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"Geometric algebra" is a set of classical geometrical techniques for defining and manipulating variable quantities. We propose that using geometric algebra may be an effective learning path for secondary students to approach analytic geometry, avoiding the "rush to symbolism" and divorce from geometry that frequently occurs. Geometric algebra is sympathetic with students' geometrical intuitions and, in the context of dynamic geometry software, it is a semi-abstract means to construct and manipulate variables, equations and inequalities without algebraic symbolism. Dragging of geometric constructions becomes a means of solving "equations", and this keys in with students' familiarity with trial-and-error techniques in the classroom. We describe some experiments carried out with 15-year old students which involved geometric constructions for squares and square roots, based on Thales' Theorem. In the former case, the students experienced a dis-equilibrium concerning the relative magnitudes of x^2 and $2x$, which they were able to resolve by conjecturing and dragging.

1. INTRODUCTION

Geometric algebra is a technique originally developed by the Ancient Greeks, nowadays considered to be something of an historical curiosity along the road of the development of algebra into its modern symbolic form (Kline 1972, Bashmakova & Smirnova 2000). We would like to suggest that geometric algebra is of more than historical interest, however, and that it may have a role to play in helping students to develop their understandings of Cartesian geometry and symbolic algebra, by introducing algebraic *ideas* before algebraic *symbols*. As Mason (1997) points out, algebra is usually developed in the curriculum as a form of generalised arithmetic, with a "rush to symbolism" that is divorced from geometry.

We have observed that students possess certain intuitions for geometry, that seem to reflect "pre-symbolic" ways of thinking. This was initially suggested to us by earlier work (see Cha & Noss 2001, 2002) where, in the context of a simple locus problem, the students seemed to spontaneously adopt a "pre-Cartesian" way of thinking about coordinates.

In the classroom, it appears that the National Curriculum tends to reinforce certain habits in students: a habit of measuring distances in any geometrical situation (eg. if asked to find a midpoint, students typically respond 'measure the length and divide it

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by 2'), and a habit of trial-and-error experiment. We have tried to harness and build on both these tendencies in our task designs. For example, the students had been taught a simple trial and improvement method for finding square roots, and we sought to echo this in geometry by setting up a construction in which the equation $x^2 = a$ could be solved by dragging to make two points coincide, at the same time as being able to look dynamically at the measured values of x, x^2 and a (see below).

2. GEOMETRIC ALGEBRA WITH DYNAMIC GEOMETRY

In geometric algebra, line segments are used to represent magnitudes, and the segments are operated on using the theorems and axioms of (Euclidean) geometry. There are, of course, four fundamental operations: addition, subtraction, multiplication and division. (Formally speaking, the field of (positive) real numbers can be constructed and manipulated through geometrical construction — see Courant & Robbins 1996, pages 120 – 22.)

We will focus here on the multiplication and division operations, which are based on Thales' Theorem: *If two straight lines are cut by three parallel lines at points A , B, C and A', B', C', respectively, then A'B' / AB = A'C' / AC* (Figure 1).

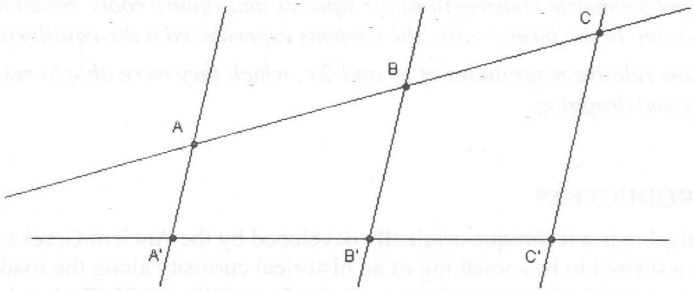

Figure 1: Thales' Theorem, $A'B' / AB = A'C' / AC$.

We will take the special case where the points A and A' are coincident, and thus we have two similar triangles ABC and AB'C'. To perform multiplication and division using Thales' theorem, as illustrated in Figure 2, it is necessary only to introduce a unit length.

One advantage of the geometric algebra approach is that it presents multiplication and division in a unified way, with their inverse relationship present in the constructions. A common misconception for students who learn arithmetical operations via counting of discrete quantities is that 'multiplication makes bigger, division makes smaller'. The falseness of this 'rule' when it is applied to real numbers can be made very clear in geometric algebra, where the constructions do not make any separation between discrete and continuous quantities (ie. integers and real numbers).

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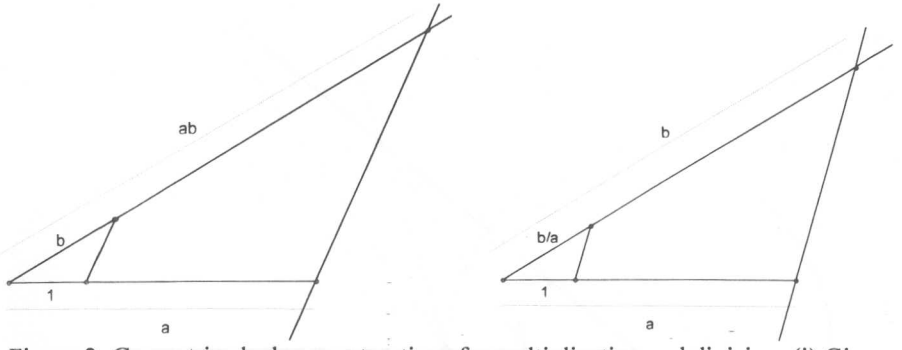

Figure 2: Geometric algebra constructions for multiplication and division: (i) Given a, b and unit segment, join end of b to end of unit, then a line parallel to this through end of a; (ii) join end of a to end of b, then line parallel to this through end of unit.

A "Dynamic" Thales' Theorem

The multiplication construction can be directly applied to the case of finding the square (or any positive integer power) of a given number — see below. In pen-and-paper geometry, to find square roots requires the introduction of a new construction (to be found in Euclid Book 2, Prop. 14; see Kline 1972, Chapter 4). With dynamic geometry, however, the squaring construction can be inverted, and the square root found by "dragging to solve" (see Section 4 below).

In general, geometric algebra makes arithmetic "semi-abstract" because of the concrete representation of numbers as magnitudes, which are manipulated concretely. Dynamic geometry enhances this characteristic of geometric algebra, particularly because dragging allows the "variable" magnitudes in a construction to be actually dynamically variable.

3. THE x^2 TASK

We will now briefly describe some experiments on geometric algebra performed with two (independent) pairs of 15-year old students, using *Geometer's Sketchpad* software.

After giving the students some familiarisation with the software, the constructions for multiplication and division were introduced using pencil-and-paper, then the experimenter showed the students (with *Sketchpad*) the construction to find x^2 from a given magnitude x, and then extended the construction to make x^3 (Figure 3).

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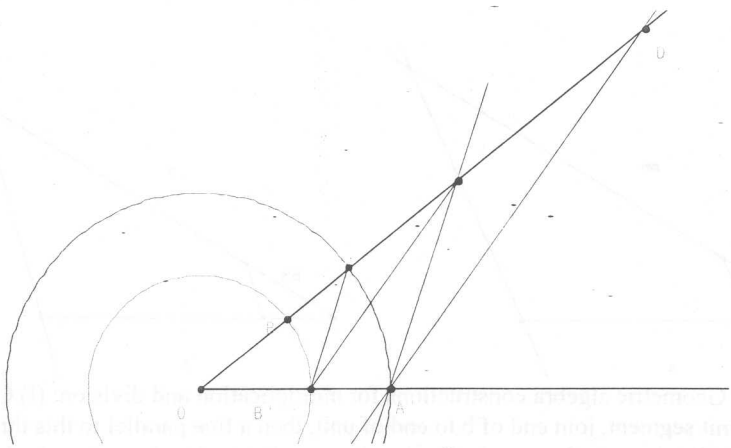

Figure 3: Construction of x^2 (OC) and x^3 (OD) from a given x (OA).

The intention with these constructions was not to use numbers for the magnitudes at the beginning, but to wait and see if and when the students started talking in terms of numbers, and only then introduce them to the construction. The students recognised that the construction should give them x^2 and x^3, however the visual appearance of the construction caused them some uncertainty, because the magnitudes had been set such that x^2 was smaller then $2x$:

> Tracy: I am still baffled about this distance being x-squared … it doesn't look like x squared, it's too small.

Tracy reveals a deeply-rooted sense that x^2 should be bigger than $2x$, despite knowing (in pre-interview) that this is not the case when x is less than 1. But in fact, for students who are in early algebra, it is well-known that the symbolic language is far from unambiguous (consider, for example, the mistake of equating $3^2 = 6$, $4^2 = 8$, etc).

The students began to explore the situation by using circles to measure off the distances $2x$ and $3x$. Then the experimenter intervened to suggest systematically comparing the magnitudes x^2 and $2x$ as the segment length x was varied. This was very effective in allowing the students to clarify their understanding:

> Shinwha: Can you find me the point where x squared becomes 2x?
>
> Naomi: Can I just experiment anyhow?
>
> Shinwha: Sure.
>
> Naomi: … the x squared has gone way up there and x is still kind of small. That means x squared is bigger than 2x.
>
> Shinwha: When?

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Naomi: When x is bigger than 1… (*Shinwha prompts to look again*) … when…when x is twice size of 1 (*making a judgement by eye*).

Shinwha: What is your conjecture from this?

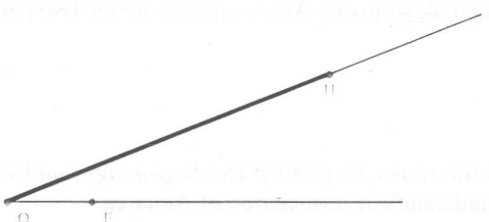
Naomi: … Something like when x is twice size of 1 then x squared is bigger than 2x. When x is twice size of unit then x squared is bigger than 2x otherwise it's smaller.

Notice that all of this was performed without the use of numbers. Also notable was the use of Sketchpad's 'Circle' construction to carry lengths around the diagram.

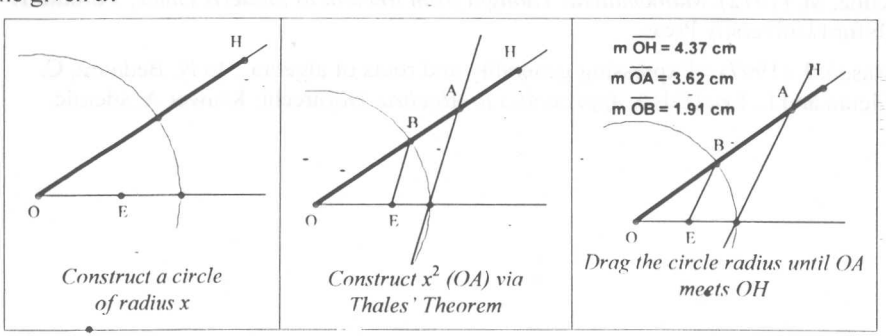
4. THE $x^2 = a$ TASK

After the squaring task, the same students were presented with the following:

The aim of this task is to use Thales Theorem to find square roots. You have a given segment of length a (OH), and a unit segment OE. Find a segment x which satisfies $x^2 = a$.



Their first attempt at a solution led them to a construction for $1/a$. The experimenter intervened to put numerical magnitudes onto the construction so the students could see that they had not found the right segment. Then, the students used some trial and improvement steps with line segments (ie. guessing a value for x and checking the value of x^2). Finally, the experimenter pointed out how dragging could also be used for trial and improvement – by simply choosing a segment 'x', constructing x^2 and then dragging the length of segment x until the end of the segment x^2 coincides with the given a:



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It was notable that the other pair of students, who had spent much more time working on the squaring task, readily adopted the dragging approach.

5. CONCLUSIONS

"Dragging to solve" could be a powerful technique pedagogically. It can be easily extended to graphical situations (ie. on conventional Cartesian coordinates, which can be connected via loci to geometric constructions), and, whereas the students' understanding of the numerical trial and improvement technique was distinctly procedural, we think there is evidence that geometric algebra may be very effective for a deeper level of understanding.

The findings described here are initial ones from a current piece of research for Shinwha Cha's PhD, being supervised by Professor Richard Noss. We think the results so far are very promising, and we are now working to devise a sequence of learning activities on geometric algebra (to include the two described here) which will be extensively tested in schools later on this year.

The Sketchpad files used in the study can be downloaded from: http://myweb.tiscali.co.uk/shinwha. Also available are the two papers by Cha & Noss listed below.

REFERENCES

Bashmakova, I. and Smirnova, G. (2000). *The Beginnings and Evolution of Algebra*. Washington, DC: Mathematical Association of America.

Cha, S. and Noss, R. (2001). "Investigating students' understanding of locus with dynamic geometry". *Proceedings of the British Society for Research into Learning Mathematics*, Southampton meeting, November 2001.

Cha, S. and Noss, R. (2002). "Designing to exploit dynamic-geometric intuitions to make sense of functions and graphs". *Proceedings of the 26th Psychology of Mathematics Education Conference* (to appear).

Courant, R. and Robbins, H. (1996). *What is Mathematics?* Oxford University Press.

Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times*, Volume 1. Oxford University Press.

Mason, J. (1997). "Expressing generality and roots of algebra". In N. Bednarz, C. Kieran and L. Lee (Eds.), *Approaches to Algebra*. Dordrecht: Kluwer Academic.

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