Partial knowledge of understandings needed for proportional reasoning

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Improving the function of multiple-choice items could be achieved by awarding credit when a person selects an option which, while incorrect, indicates that they have some knowledge of the tested concept. To support the identification of what might constitute partial knowledge, Rasch Measurement Theory was applied to students' responses to items on ratios and fractions. Based on previous research, the answers to individual parts of questions were classified as being fully correct, partly correct or incorrect and scored accordingly. Examination of the category probability curves produced by the RUMM2030 software during the analysis indicated for which items the award of partial credit was justified. This study suggests that the identification of items for which partial credit is warranted requires both qualitative and quantitative analysis.

Keywords: partial knowledge; proportional reasoning; item difficulty

Introduction

Even though they are perceived to be problematic by some educators, multiple-choice (MC) items predominate in high profile assessments of mathematical understanding. One problem is that the amount of information about student learning that is collected from each item is limited. MC items generally contain one correct answer which receives a score, and three or four incorrect alternatives for which the score is zero. To be an effective item the correct answer should not be obvious to candidates and incorrect options should be plausible. This indicates that there are likely to be some elements of the knowledge required for that item addressed in the incorrect options.

This study is one component of a larger project in which the focus is to identify ways by which multiple-choice items can be used to collect more information about student attainment. In this part of the project the aim is to identify partial knowledge of concepts for the creation of incorrect options for which some credit can be given. Students selecting such incorrect options are likely to have greater knowledge of the concept than students selecting the other incorrect options but not as much as those who select the correct option. According to Hodgen, Brown, Coe and Küchemann (2012), these errors may develop as understanding increases and by examining the responses to an item in relation to student performance on a test, such errors may be identified.

Proportional reasoning was chosen as the context for the project as it is an important concept for students in lower secondary. It has been described by Watson, Jones and Pratt (2013) as one of the seven key concepts which spread across and connect the many aspects of the mathematics curricula. It is "not taught explicitly" but "valued implicitly" (p.14). The skills and understandings necessary for the development of sound proportional reasoning can be identified from the author's experience of teaching this topic and from research by Lamon (2005) and Tjoe and de la Torre (2014). Fractions, decimals, ratio, rates, percentages, scaling and relationships between variables were considered for the investigation.

Partial knowledge

Identification of what constitutes partial knowledge in a concept involving proportional reasoning is challenging. Using addition instead of multiplication to solve problems involving proportions has been widely reported in the literature (Karplus, Pulos & Stage, 1983; Misailidou & Williams, 2003; Noelting, 1980; Tourniaire & Pulos, 1985) and has been described as the use of additive strategies or as additive thinking. When there is a proportional relationship between two variables students used the difference rather than the scale factor to determine missing values. Misailidou and Williams (2003) concluded that students who made this type of error are more capable than those who make other types of errors and suggested that additive thinking may even be a step along the way in their development.

Another type of error involves the lack of consideration of the proportional nature of the question and only part of the information supplied is used to solve the problem. This could occur for the item described by Hart (2004) where the student is given that for every 1 part of mercury there are 5 parts of copper and for every 3 parts of tin there are 10 parts of copper. Ignoring the amount of copper, the student would incorrectly conclude that for every 1 part of mercury there are 3 parts of tin.

Other types of partial knowledge might exist with the recognition of the operations needed to solve the problem but an inability to complete the computation. Given the area of a rectangle and asked to determine the length, students may recognise the need to divide the area by the length but if the numbers are unfamiliar or fractional the students may identify the computation process but not the answer.

At times, students know an approximate but not exact value for the answer. The question about the length of fishfingers being fed to eels depending on their length (Piaget & Inhelder, 1951) is one where this could occur. Given that an eel which is 25 cm long is fed with a fishfinger that is 10 cm long, students are asked to determine the length of a fishfinger for an eel that is 15 cm long. Responses of 7, 8 or 9 cm indicate that the student has the idea that the fishfinger has to be more than 5 and less than 10 cm but was unable to determine the correct response. For such items students are able to provide reasonable estimates.

Data analysis

The responses of students from Years 7, 8 and 9 (n=4950) in tests on some of the skills necessary for the development of sound proportional reasoning, including ratios and fractions, were analysed using Rasch Measurement Theory (Rasch, 1960). The test questions, which were all of the constructed response type, were initially developed for the Concepts in Secondary Mathematics and Science (CSMS) Study and later updated as part of the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) Study (Hodgen, 2016). The data collection occurred in 2008 and 2009 and was part of Phase I of the ICCAMS project funded by the Economic and Social Research Council in the United Kingdom.

The application of Rasch Measurement Theory to the students' responses allows the identification of a conjoint measurement scale for estimates of item difficulty and person ability. It is also possible to allocate different scores for different responses and to identify the resulting levels of difficulty associated with these different responses. The software program Rasch Unidimensional Measurement

Model, RUMM2030 (Andrich, Sheridan & Luo, 2013) was used to apply Rasch Measurement Theory to the students' responses for this project.

From the 17 test questions, 38 items were identified. All items were used in the analysis to provide the most accurate estimates possible for student proficiencies. Missing responses were recorded as if the students had not been presented with the item rather than as being incorrect. Polytomous scoring was used for the 30 items deemed to be worth partial credit with 0 for an incorrect response, 1 if the answer was incorrect but indicated the student had some knowledge of the concept and 2 for a fully correct response. Responses for the remaining eight items were scored dichotomously by awarding 1 if correct and 0 otherwise.

Partial knowledge was allocated to one of the four different types described earlier and identified here as (a) additive thinking, (b) incomplete solution, (c) proportion ignored and (d) acceptable estimate. Further examples are provided. Additive thinking was identified in the question asking students to identify missing numbers for equivalent fractions, for example, entering 5 for the following missing number question, $\frac{6}{8}=\frac{3}{}$. Incomplete solutions were provided when students showed the operations necessary to determine the solution or showed an answer that was an interim value in determining the answer. When students used only the first numbers in ratios to make comparisons or gave 24% as the answer for 24 out of 800, partial knowledge was recognised even though they had ignored the proportional nature of the question. Acceptable estimates were identified and partial knowledge accredited when the students gave an answer less than a half when asked how much cream is needed for 6 people given half a pint is needed for 8 people.

Findings

From the analysis, ordered thresholds were present in nine of the thirty items scored polytomously. This could be recognised by examining the category probability curves produced by the RUMM2030 software as shown in Figures 1 and 2. These curves show the relationship between the students' achievement on the test (Person Location measured in logits) and the probability of obtaining 0, 1 and 2. For Item 16 as shown in Figure 1, the threshold of achievement at which students are more likely to receive a score of 1 than 0 (about 0.3 logits) is lower than the threshold where students are more likely to receive a score of 2 than 1 (about 3.2 logits). The probabilities of higher scores for the item correspond with increasing achievement on the test.

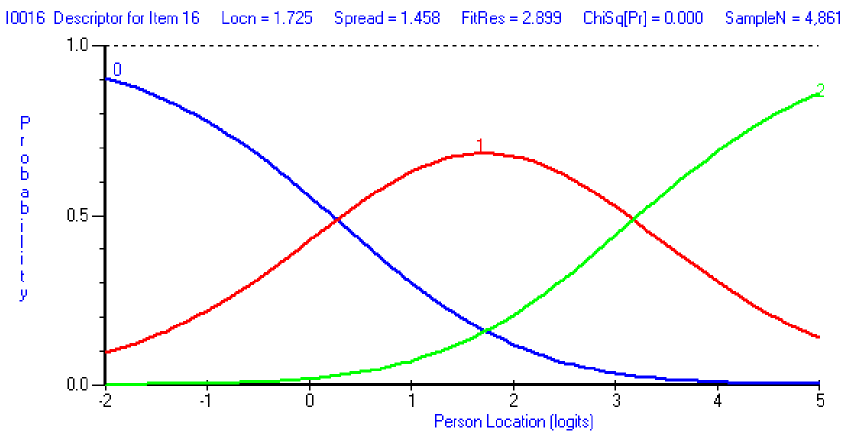

Figure 1. Category probability curve for Item 16

For Item 16 the students were given the diagram for a right angle with sides of 3 cm and 2 cm. They were asked to work out the length of the shorter arm in the second diagram where the longer arm was 5 cm and the diagram had the same shape. Partial credit of one mark was awarded for an answer of 4 cm (additive thinking) and the correct answer scored 2 marks.

Similar scoring for additive thinking was used in Item 8 and the category probability curves are shown in Figure 2. In this question, eels were fed with sprats according to their length and students were asked to enter the number of sprats for an eel measuring 15 cm when given that an eel measuring 10 cm was fed 12 sprats. Partial credit of one mark was awarded for an answer of 17 (additive thinking) and the correct answer of 18 scored 2 marks.

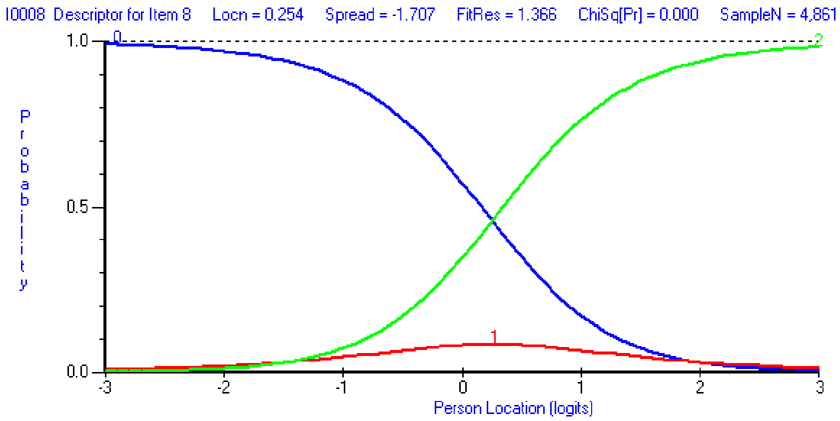

Figure 2. Category probability curve for Item 8

The thresholds are ordered for Item 16 and disordered for Item 8 where having greater ability on this test is not aligned with the probability of obtaining higher scores. The probability of scoring 1 is not higher than the probability of scoring 0 for students of medium ability and at no point on the continuum of achievement is the probability of scoring 1 the greatest. By comparison, scoring 1 in Item 16 has the highest probability for persons of medium ability in the estimated range of Person Location from 0.3 to 3.2 logits. The scoring for Item 8 is not functioning properly (van Wyke and Andrich, 2006).

Of the thirty items for which an incorrect answer was awarded partial credit in this analysis, eight had ordered thresholds. For these, partial credit was given for the use of additive thinking in four of the items and for the other items credit was given for identifying an answer within an acceptable range, incomplete scaling or for absolute value thinking (ignoring the proportion). The proportions of students who were correct (s=2) and who received partial credit (s=1) are shown in Table 1.

Item	5	15	16	17	18	19	20	21
s=2	0.12	0.06	0.03	0.22	0.20	0.08	0.09	0.11
s=1	0.41	0.15	0.27	0.44	0.24	0.20	0.39	0.29

Table 1. Proportion of students with complete or partial scores on items given partial credit.

To generate more accurate estimates for item difficulty, further analyses were conducted. In the second analysis the items with disorderd thresholds were re-scored dichotomously and then all items were checked for response dependence. This statistical dependence occurs when the correct answer to one item is necessary for, or a clue to, the correct answer to a subsequent item. It was expected in these data because parts of the questions were used as separate items. Dependence was identified in eight items by the high correlation between the residuals and the items were eliminated in the third analysis. The estimates of item thresholds and the distances between them were then deemed to be more reliable and were used to identify a more accurate ranking of item thresholds as well as the distance between thresholds. As a result, it was possible to compare and rank items in order of difficulty.

Discussion

Most of the items believed to be worth partial credit did not function as such and were best scored dichotomously. These included all the items on fraction equivalence. This suggests that additive thinking about the relationship between the numerator and the denominator is not a misconception that develops as the students' ability to understand fractions increases. For the two challenging items on the manipulation of fractions, partial knowledge was not confirmed in the analysis. It is possible that the students found the identification of the necessary operation more difficult than the execution of it and this could explain the failure of polytomous scoring in that item.

All four items relating to operations with percentages functioned better as dichotomous items and the expected partial knowledge for additive thinking and incomplete solutions in these items was not associated with the development of those concepts for students. It is suggested that as students learn about percentages any misconceptions that develop are not linked to knowing that percentages represent proportions. Awarding partial credit for additive thinking in all the items relating to food for eels as described earlier (Piaget & Inhelder, 1951) was not justified. It is possible that the presentation of numbers as easily-recognised simple multiples of 5 (5, 10 and 15), triggered multiplicative thinking rather than additive thinking.

Items working successfully when credit was given for answers that were not fully correct came from all four different types of partial knowledge. Where partial knowledge was given for incomplete solutions, ignoring the proportion or for reasonable estimates, there were two few of these items to identify any patterns. The success of the four items associated with additive thinking may be associated with the nature of the ratios in that the scaling was not integral, unlike that of the items regarding eels.

It has been possible in these analyses to identify which items work best with polytomous scoring and those working best with dichotomous scoring. It has also been possible to re-score items as dichotomous when not working otherwise. This has resulted in the calculation of better estimates of item difficulty and person proficiency.

Conclusion

The ability to recognise the justification for awarding partial credit can provide important information for the development of multiple choice items. Scoring multiple choice items with partial credit can be adopted without adding to the test demands on the student. It will, however, require greater expertise from the authors of MC items but should allow items to function more efficiently and more accurate measures of student achievement to be estimated.

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