Implications of Giaquinto's epistemology of visual thinking for teaching and learning of fractions

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In this paper, I will present some implications of Marcus Giaquinto's ideas about visual thinking and its epistemology when combined with Toulmin's layout of an argument. This is the result of an ongoing effort to discuss, from a theoretical perspective, some issues that emerged from my Ph.D. research about teaching and learning addition and subtraction of fractions to low achieving students. My claim is that visual representations can be effective for low achieving students when teaching is focused on a carefully chosen model and time is given to students to fully use it.

visual representations; fractions; reasoning

Introduction

The goal of my Ph.D. research project is to investigate how low achieving students learn fractions through an approach aimed at building their knowledge using visual representations. I observed several lessons in a British secondary school and, based on them, designed a series of lessons covering topics from equivalence to addition and subtraction of fractions for students placed in low sets in Year 8 and 9. The lessons were based on the rectangular area model initially realized through card board manipulatives and later through diagrams. Three teachers enacted these lessons with one group each. Data was collected through 'within-class clinical interviews', which consist of a) audio recordings, taken during lessons, of interactions between me and a student usually initiated by a 'why' or 'how' question, b) notes taken by me right after each interaction and c) students' worksheets.

During my data collection and in the beginning of my data analysis, I identified episodes of low-achieving students, that were not used to justify their answers beyond the description of steps taken to obtain the final answer, being successful at constructing mathematical arguments by anchoring their reasoning on visual representations.

The hypothesis I defend here is that the nature of the visual representations opened up a way for the students to argue mathematically. The sections below are an attempt to articulate the ideas regarding visual thinking by the philosopher Marcus Giaquinto (Giaquinto, 2007) with ideas related to reasoning using Toulmin`s layout of an argument (Toulmin, 1969) in order to support my hypothesis from a theoretical perspective.

Visual thinking

'Visual thinking in Mathematics' (Giaquinto, 2007) is a philosophical study that articulates empirical results from cognitive sciences with theoretical arguments in order to discuss the epistemological role of visualization in concept acquisition and knowledge formation in mathematics. The general aim of the author is to raise the status of visualization, often seen as secondary or complementary both in formal and school mathematics.

According to Giaquinto (2007), visual experience can lead to concept acquisition and creation of new knowledge. The author defines concept acquisition by stating that a concept is acquired when one is able to form a thought involving it. For instance, to acquire the concept of uncle, a person has to form a thought such as 'Uncle is the brother of a father or mother'. In this case, brother, father and mother are constituents of the thought and had to be acquired beforehand. Giaquinto's argument is that previously acquired concepts and empirical evidence are not the only options for constituents of a thought, visual experiences can also play that role.

For instance, a person could form a thought involving 'square' using the perceived properties of symmetry, parallelism and straight edges as constituents. Since empirical evidence points that perception of symmetry, parallelism and straightness are innate or very prematurely developed in humans, a person can form that thought by purely visual means (obviously there is no need of using the particular word 'square'). This way, the person would be acquiring a visual concept of square which can, in the future, be linked to the actual word square and to the geometrical concept of a square (entailing the precision and language demanded by mathematics).

This mechanism can be expanded beyond basic concept acquisition to knowledge creation in general. In this case, Giaquinto reinforces the importance of the transformations applied to concepts acquired visually or associated with a visual image as opposed to the importance of basic visual skills. An example of knowledge creation through visual means would be the famous episode of Meno's slave, originally described by Plato, coming to believe that the area of a square obtained by connecting the midpoints of each side of a given square is half of the area of the latter (Giaquinto, 1993). The result is achieved by transforming the larger square through folding its corners over the sides of the internal square. According to Giaquinto (1993), there is no need of extra definitions or deductions since the result can be perceived visually by the overlapping of areas.

This way of acquiring new knowledge can be seen in opposition to empirical

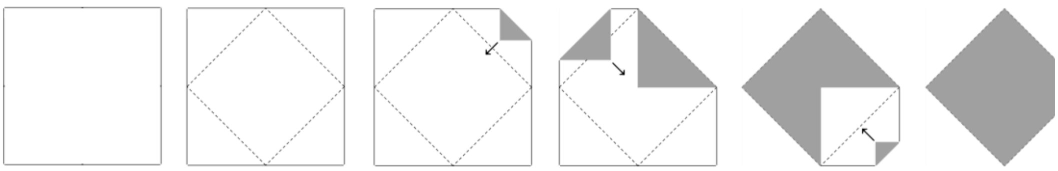

Figure 1: Folding a square into the square formed by segments connecting adjacent midpoints.

(based on collecting samples of entities that do and do not fit into a particular concept) and analytical (based on unpacking definitions and conducting logical deduction) ways. Giaquinto does not state that the visual route is superior in any sense, only that it is a reliable, valid and qualitatively different way of acquiring mathematical knowledge.

Reasoning and argumentation

The second part of my argument in this paper is based on Toulmin's (1969) layout of an argument. Although his ideas were developed for use in a very broad fashion, they are extensively used in science education and there are authors also applying it to mathematics education.

Toulmin (1969) proposes that an argument can be divided into four elements: conclusion, data, warrant and backing. Conclusion is the final claim, the answer, that is considered correct by the reasoner. Data offers support for the conclusion and typically emerges when "how" questions are asked. Warrant is about the explanatory relevance of the data, it elicits the reasons why the provided data explains the conclusion and it emerges from "why" questions.

Before moving to the fourth component, it is important to clarify that a given data or warrant may be further questioned. In this case, they become the conclusion of a new argument and the reasoner has to provide further data or warrant to support it.

The fourth component, backing, is the ultimate warrant, the one that does not admit further questioning. It is the basis on which the arguments are being built. A backing is constituted of shared knowledge between the person arguing and the person questioning and the acceptance of a warrant as backing is a social construction (Yackel & Cobb, 1996).

In this paper, I will focus on warrants and backings without further concern about differentiating them, since Toulmin also recognizes that this may not be simple and usually does not affect the understanding of an argument.

Warrants

Following Toulmin's ideas, Yackel and Cobb (1996) propose that it is possible to identify two types of warrants when it comes to arguments supporting mathematical tasks: social and mathematical. The first is based on the authority of somebody else, such as a teacher, while the second is based on properties of mathematical objects.

It seems reasonable to draw a parallel between this idea and Hewitt's (1999) distinction between arbitrary and necessary knowledge. He defines the terms by saying that "something is arbitrary if someone could only come to know it to be true by being informed of it by some external means" (p. 3), while "[t]hose things which are necessary can be worked out" (p. 4). The focus of his paper is that teachers should be aware of that distinction and adopt different teaching strategies for arbitrary and necessary topics in the curriculum. Although these concepts are not explicitly related to warrants by Hewitt (1999), I would say that his ideas can be translated in terms of warrants. On one hand, for arbitrary knowledge, although teachers may adopt memory aids or use history of mathematics while teaching them, the warrants will be ultimately authoritative. On the other hand, necessary knowledge "are properties which can be worked out from what someone already knows" (p. 4), which means that its warrant can be based on prior knowledge.

A common point made by these authors is that warrants of the latter type should be prioritized. For instance, Hewitt (1999) argues that "mathematics does not lie with the arbitrary, but is found in what is necessary" (p. 5). But here comes an issue: what can be done when teaching low achieving students who, in general, may not hold sufficient mathematical knowledge?

My proposal is that, analogously to what Giaquinto proposed for mathematical knowledge acquisition based on visualization, we should recognize that mathematical knowledge, as well as mathematical arguments, can be built on elements that do not necessarily come from school mathematics.

For instance, Mack (2001) suggests that informal knowledge can be an option. More specifically, she argues that multiplication of fractions can be taught using students' informal knowledge of partitioning and shows it through a case study. In terms of Giaquinto's ideas, Mack's proposal uses students' informal knowledge as constituents to form new thoughts regarding multiplication of fractions and the inherent properties of partitioning to create new knowledge.

Mack's (2001) assumptions are that informal knowledge of partitioning: 1) is available to students, and 2) can be a good 'building block' to construct new knowledge related to this topic. Analogously, Giaquinto(2007) uses research in cognitive science to show that visualization is available to everybody (assumption 1) and then argues throughout his book that it can be a very effective 'building block' for certain topics (assumption 2), such as geometry and arithmetic, but not as effective for others, such as real number properties.

Mack (1995) also recognizes that informal knowledge on partitioning was not so successful to teach addition and subtraction of fractions. Since she expects that informal knowledge on partitioning is quite prevalent, it seems that the reason for this result comes from partitioning not being an effective 'building block' for the topic.

Although my data analysis is not finished, I believe my data will offer empirical evidence that both assumptions are true for visual thinking, when used under certain conditions, for teaching and learning fractions.

Fractions and visualization

Let us consider an example on equivalent fractions. I once taught fractions to secondary students using a textbook which adopted an approach that relied heavily on decimals: a fraction would be defined as the result of the division of numerator by denominator. Within this approach, to explain why 1/2 is equal to 2/4, I would ask students to compute 1 divided by 2 and 2 divided by 4 and compare the answers. In general, I would use the property that says that 'quotient remains the same if dividend and divisor are multiplied by the same number' to justify equivalent fractions. In that particular occasion, the approach was successful. However, it seems bound to fail with a group of students which struggled with arithmetic properties.

Alternatively, the same result could be explained by means of the following diagrams.

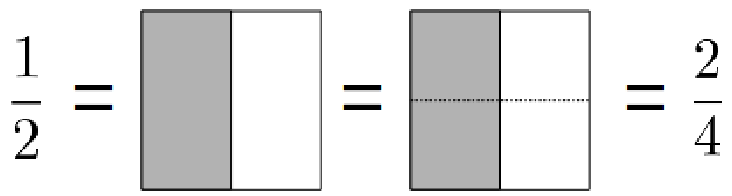

Figure 2: Equivalent fractions by visual means

Note that the interpretation of what is being shown in the diagrams depends on some visual skills, such as perception of area (not the formal concept, involving computations) and composition and decomposition of shapes.

Although I cannot state that these skills are readily available to any student, I would argue that it is possible to achieve that by certain choices regarding models and materials to be used in the classroom.

During the lessons I designed for my data collection, I adopted the rectangular area model materialized through cut-outs because they embed the properties and operations that are necessary to build the main concepts related to fractions in a very accessible way: composition and decomposition of shapes can be achieved by juxtaposing the cut-outs and preservation of area by overlapping them. In other words, the use of cut-outs to represent fractions in the rectangular area model makes the structure necessary to operate with fractions very transparent.

Using this approach, the teachers participating in my research could avoid teaching based on authoritative warrants, typical when teaching is based on procedures without knowledge of the reasons behind them. Also, this approach allowed them to avoid the frustrating feeling of trying to build the new knowledge of topics on which the students are not fluent, as it could have happened with an approach based on decimals.

Additionally, it is possible to point out another two effects of an approach strongly based on visual representations. The first refers to the development of visual skills that are known to be strongly correlated to general achievement in mathematics (Verdine, Golinkoff, Hirsh-Pasek, & Newcombe, 2014). The second refers to the reduction of the reliance on language that may be affecting negatively students from disadvantaged backgrounds (Gates, 2015). Although I consider both effects very relevant, it is beyond the scope of this paper to discuss them in depth.

Implications beyond fractions

Giaquinto's main point is that mathematical knowledge can be validly built on visual thinking. As a consequence, I would argue that arguments, which can be seen as external manifestations of mathematical knowledge, could also be based on this 'building block' and that this approach can be particularly effective for low achieving students.

Nevertheless, this is not a straight forward job. Although experimental research shows that some visual capacities, such as recognizing symmetry, are innate or at least develop at a very early age, Giaquinto (2007) points out that visualization also develops through the experience of seeing and by learning where to focus attention.

Bearing this in mind, I suggest that, in order to build mathematical knowledge on visualization, teachers, textbooks and designers should focus on a small number of powerful visual representations. This is being intentionally phrased in opposition to a common assumption that knowing multiple representations for the same topic is necessary for conceptual understanding. I do not deny that, for some topics, there are several representations that a student is expected to know by the end of her/his school life, but I see this as a curricular issue instead of a decision made to promote or facilitate teaching and learning.

Furthermore, it is crucial to choose visual representations that embed the relevant properties and transformations in a way that is accessible for the students and, sometimes, this characteristic does not belong to the visual representation itself, but to how it is materialized. Consider, for instance, the case of the rectangular area model for fractions. It can be manifested through cut-outs in different sizes according to the fraction each piece represents, implicitly embedding a constant size for the unit. Another possibility is to use acrylic shapes all with the same shape and different colours, so that a representation of 1/3 and 1/4 can only be achieved by using units with different sizes (therefore, the fractions would not be comparable) or by choosing a suitable size (a common multiple) for the unit. A similar issue takes place if one chooses diagrams instead of manipulatives: drawing them on a graph paper may lead to the same limitations as the acrylic shapes example, while drawing them on blank paper removes these limitations but adds the challenge of actually drawing the diagrams more freely by hand.

During my research, I adopted the rectangular area model manifested at first through cut-outs with different sizes according to the fractions they represented. Then, after some lessons, the approach moved towards diagrams. I started with the cut-outs because, as discussed previously, they embed the properties and transformations necessary to achieve the goals established for the lessons (addition and subtraction of fractions) very transparently. The movement to diagrams was made because they are not limited by the availability of pre-cut shapes and, although I consider them less transparent, I expected the students to be able to transfer the properties and transformations from the cut-outs to the diagrams, given they were both based on the rectangular area model.

Finally, it is important to allocate time for the students to interact with the visual representations, going beyond merely reading and writing, but also understanding what can be done to them.

What I want to reinforce here is that the choice is not simple and every detail related to which visual representation is chosen, how it is presented to the students and how they can actually use the representation is crucial to enhance the possibility of learning new mathematics and build sound arguments through visual thinking. In spite of that, this type of approach can be especially beneficial to low achieving students.

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