Working Group report: A brief history of functions for mathematics educators

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Despite the words: ‘Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history’s most intriguing problems.’ in the Purpose of Study section, the new English mathematics curriculum alludes only to Roman Numerals in the primary programme of study, and there is no further mention of historical or cultural roots of mathematics in the aims, or in the programmes of study. In contrast, the increased expectations for lower and middle attainers in the new curriculum challenge teachers to make more mathematics accessible and memorable to more learners. The history of mathematics can provide an engaging way to do this. There are also many opportunities in post-16 mathematics. Further to our recent articles on quadratic equations and trigonometry, we use functions to illustrate some of the ways that history of mathematics can enrich teaching of this topic.

Keywords: history of mathematics; functions; BSHM

Introduction

In the 2014 National Curriculum for mathematics (DfE, 2014) “using functions” is a sub heading of algebra for Key Stage 3 that includes the solution of equations and graphical representations of relationships between variables. In Key Stage 4, all students are expected to be able to “find the inverse functions for familiar one-to-one functions” but only students working towards higher tier GCSE are expected to understand and use function notation, derive composite functions and reflect and translate functions. In Upper Key Stage 2 representing missing number problems algebraically and generalising rules of arithmetic (e.g. $a + b = b + a$) and rules for linear sequences are included, but the word ‘function’ is not used. The Upper Key Stage 2 curriculum formalises earlier primary work on recognising patterns, counting, generating sequences and finding missing numbers in calculations. Formal work on functions is only introduced at A level where notions of domain, range, and continuity are met as foundations for calculus. The history of functions can help teachers to appreciate why this topic may appear very abstract for learners but how its roots might inform teaching that develops strong foundations for future development.

In our paper on the history of quadratics (Rogers & Pope, 2015) we set out the rationale for using the history of mathematics in education. In this short article, we introduce some key resources and explore the development of associated mathematical ideas that are suitable for the classroom (Watson, Jones, and Pratt (2013).

Defining a function

Leibnitz is generally acknowledged as the first to use the word function to describe the tangents to a curve (1692). Leibnitz and Johan Bernoulli used ‘function’ to refer to the ordinates, tangents and radii of curvature of a given curve. Euler (1707-1783) in

Introductio in Analysin Infinitorum (1748) introduced the concepts and methods of analysis and analytic geometry needed for a study of the calculus. The allowed operations are: addition, subtraction, multiplication and division together with roots, exponentials, logarithms (as exponents), trigonometric functions (as numerical ratios), derivatives and integrals. 'Functions' were algebraic (polynomials) or transcendental (e^x, or sin*x*) single-valued where each element of the domain maps to a single, well-defined element of its range or many valued; and explicit ($y = 2x + 1$) or implicit ($x^2 + xy = y^2$).

> A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities (Euler 1748, p.72)

This notion of function is not the idea of dependency or the correspondence of values, but expresses a relationship explicitly in the form of an equation in ordinary symbols and operations.

A major belief among 18th century mathematicians was: *If two analytic expressions agree on an interval, they agree everywhere.* However, mathematicians realised that any curve drawn free-hand in a plane can determine a functional relationship that *may not be representable in an ordinary analytic form*. This perception implied that *algebraic equations are not sufficient* to express the general function concept. 'Function' was defined as 'an analytic expression' (meaning a formula of some kind) – but this was not clearly defined until the late 19th century, after new concepts had been introduced by Dirichlet, Riemann, Weierstrass and Dedekind with a new theory that *clarified the role of algebra, trigonometry, the real line continuum and rules about the convergence of series.*

The problem of the vibrating string

As an example of the development of functions we consider the problem of the vibrating string. The initial questions about this problem go back to Mersenne (1588-1648), although it is worth noting that the Pythagoreans (6th Century BCE) knew that the frequency of a note was inversely proportional to the length of the string.

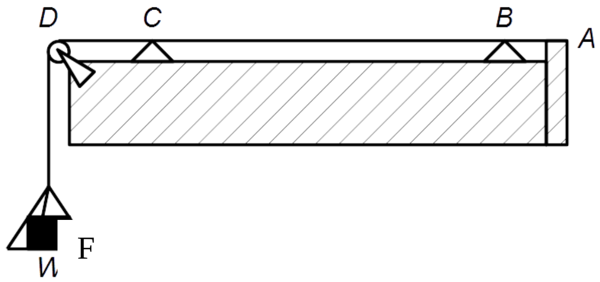


Figure 1. The vibrating string

In *L'Harmonie Universelle* (1637) Mersenne brought together the classical knowledge about music and described his experiments with a stretched string, to propose what are known as Mersenne's Laws:

1. The frequency of the note is *inversely proportional to the length* of the string (L).

$$f \propto \frac{1}{L}$$

2. The frequency is *proportional to the square root* of the stretching force (F).

$$f \propto \sqrt{F}$$

3.The frequency is *inversely proportional to the square root of the mass* (W) *per unit length (W)*

$$f \propto \frac{1}{\sqrt{W}}$$

In school physics these can be combined as Mersenne's Law: $f = \frac{1}{2L}\sqrt{\frac{F}{W}}$. Mersenne discovered the connection between musical pitch and frequency of vibration and he also realised that a number of different vibrations (the harmonics) may be happening at the same time.

By the beginning of the 18th century, the problem became probably one of the most outstanding in the history of mathematical physics:

> A uniform elastic string is stretched between two fixed points A and B at a distance *l* apart and put into small horizontal vibrations. Using the Cartesian system with AB as the *x-axis* and a horizontal line through A perpendicular to AB as *y-axis*, attempt firstly to find an equation to represent the motion in terms of *y* as a function of *x* and of time *t*, and then solve it to find an explicit expression for *y* as a function of *x* and *t*.

Both Johan Bernoulli (1667-1748) and Euler (1707-1783) were responsible for originating the study of the solution of ordinary differential and partial differential equations and Jean le Rond d'Alembert (1717-1783) discovered an equation representing this motion and published it in 1747 where the constant c represents the nature (e.g. density) of the material:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial z^2}$$

This was never called into question at the time, but the generality of the competing solutions from other mathematicians led to a lengthy controversy involving the whole of 18th century analysis: the theory of functions, the role of algebra, the real line continuum, and the convergence of series; as well as the physical interpretation of a string in motion. This was the period of the development of 'Rational Mechanics' where Bernoulli and Euler, were keen to develop Leibniz's calculus and its applications to the mechanics of rods, beams, fibres, membranes and many other such materials, including the vibrating string. During this time *new functions were being developed from the results of physical experiments*.

In the late seventeenth century there were two scientific schools of thought in France. René Descartes, in his *Discours de la méthode,* believed that everything that was true could be found through careful reasoning. However, Isaac Newton (supported by Voltaire) *believed that only by experimentation and observation could one prove hypotheses to be "true."*

It was generally recognised at the time that musical sounds (the visible vibrations of a string) are composed of fundamental frequencies and their harmonic overtones. This convinced Bernoulli that the solution to the vibrating string problem must be given by a combination of sines. So an arbitrary function *f(x)* can be re-presented on an interval (0, *l*) by a series of sines:

$$y = U \sin\frac{x}{a}$$

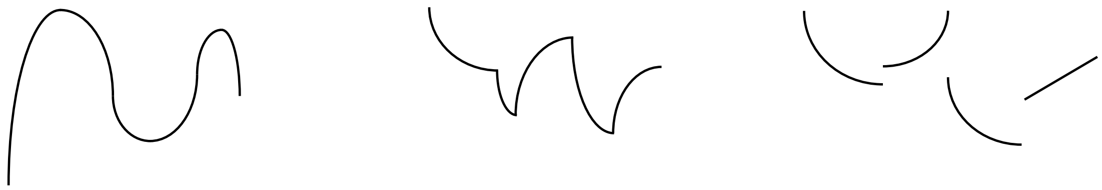
where *x* is the incremental unit of length of an element defined on the interval, *l* .

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

However, the shape of the wave varies with regard to time, so the equation becomes

$$y(x,t) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Johan Bernoulli was Euler's teacher, and by the beginning of the 18^{th} century it was generally accepted that 'analytic functions' were described by combinations of polynomials, powers and roots etc., as described above (p.2). But there were assumptions about the limits of sequences, convergence of series, and what was meant by a 'continuous' function.



Euler "continuous"
Modern "differentiable"

Euler "discontinuous"
Modern "continuous"

Euler not recognised
Modern "discontinuous"

Figure 3. General description of recognised 'analytic' functions about 1750[2]

Joseph Fourier (1768-1830) was involved in the French Revolution and later appointed to assist Lagrange and Monge at the Ecole Polytechnique in 1795. He took part in Napoleon's expedition to Egypt, and became involved in public life. He was principally a physicist, elected to the Academie des Sciences in 1822, and in the same year presented his revolutionary study *Theorie Analytique de la Chaleur* (Analytical Theory of Heat). He studied the temperature distribution of heat, ν(t, x, y), across a series of different metals and realised that it satisfied the partial differential equation:

$$\frac{\partial v}{dt} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

By separating the variables and integrating, with appropriate boundary conditions, he saw that the general form of solution involved a sum of exponential and cosine expressions such that:

$$\nu = ae^{-nx}\cos ny$$

Which, over a sequence of boundaries led to one solution as a square wave approximated by increasing the number of terms of the trigonometric series (Herivel, 1975).

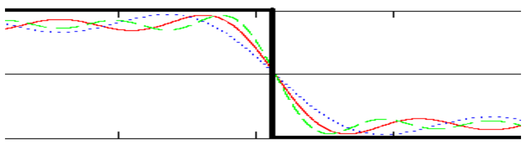


Figure 2. Temperature distribution around a boundary as a function of the number of Fourier coefficients

Immediately this led to a serious controversy:

(a) Fourier, the physicist had invented a *new kind of mathematical result from experimental evidence.*

(b) He had produced a general solution for all equations of this type, by claiming that a solution could be *built up piecemeal from non-analytic representations*

[2] This sketch is adapted and modified from Grattan-Guinness (1970) page 7.

(c) And that a solution could be found by *focussing on specific intervals of the variable.*

Fourier challenged what was understood by convergence and limits of series, and what exactly was meant by a *continuous* function.

In 1826 Cauchy was not successful in showing that Fourier's series was convergent, but Dirichlet repeated Cauchy's method using limited conditions and introduced the concept of 'boundedness' on an interval. Following this, Riemann defined necessary and sufficient conditions for particular functions to be integrable. This period shows the typical procedure in testing hypotheses in mathematics by slow modification of the conditions on specific types of function and achieving moderate successes step by step. Dirichlet and Riemann showed that in principle, Fourier's series could represent a discontinuous function, while new results came from a different area of mathematics: the work of Weierstrass on the property of uniform convergence over an interval, and Dedekind and Cantor's work on the definition of real numbers, brought assurance of the theoretical reliability of Fourier's assertions about his series, thus clarifying the rationale for the modern definition of a function. For example, MacLane and Birkhoff (1967) define a function thus:

> For given Sets S and T, a function *f* with domain S and codomain T assigns to each element s I S of the domain an element *f* (s) I T of the codomain. This element *f* (s) is called the value of the function at the argument s …. (p. 4)

Today, we understand functions as relations in which the value of a variable is dependent on one or more other variables. Particular values of the independent variable generate one and only one value of the dependent variable. Learning about functions and using them as tools for modelling situations is a long process which begins in primary school with simple equations and elementary representation of data and the use of one to many, one to one, and many to one mappings in different contexts. Throughout secondary school students need to coordinate algebraic, graphical and numerical data, often from scientific experiments or statistical investigations. Trigonometry employs algebraic manipulation in bringing together geometrical conceptions and angle measurement. These important ideas are emphasised by Watson, Jones and Pratt (2013). Many relevant original texts with their translations are available in Stedall (2008).

The use of individual data points from physical experiments to investigate a possible relationship between a process and its outcome raises questions about the manner of representation and the meaning of a line that may be drawn to join individual points on a graph.

Having some acquaintance with the history of mathematics shows us how the modern sophisticated concept of function grew from the vital contributions of many mathematicians and scientists.

Context

This short article arises from the BSRLM History of Mathematics working group as part of its preparation for an Anglo-Danish History of Mathematics in Education conference held at Bath Spa University, 22-24 August 2016.

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