Using context and models at Higher Level GCSE: adapting Realistic Mathematics Education (RME) for the UK curriculum

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Since 2003, staff at Manchester Metropolitan University (MMU) have been involved in a number of projects related to Realistic Mathematics Education (RME). This originally involved trialing materials with 11-14 year olds and then, in collaboration with the Freudenthal Institute, writing materials for Foundation level GCSE. In 2012, these materials were published by Hodder Education as a series of books called *Making Sense of Maths*. Classroom trials of the original materials showed an increased willingness of students to discuss and engage with their mathematics, and to attempt to 'make sense' of what they were doing rather than simply to try to remember taught procedures. The results from the trials led, in 2009, to a further project designing materials for Higher level GCSE. The focus here is on the development of these materials, and how we have interpreted the original design principles of RME for UK schools. In particular, we focus on the use of context, the notion of progressive formalisation and the use of models. We provide excerpts from the materials that exemplify these principles and discuss the issues for teachers attempting to integrate this approach into an examination driven curriculum.

Keywords: secondary, context, models, formalisation, teachers.

Background

The Freudenthal Institute, University of Utrecht, was set up in 1971 in response to a perceived need to improve the quality of mathematics teaching in Dutch schools. This led to the development of a research strategy and to a theory of mathematics pedagogy called 'Realistic' Mathematics Education (RME) which is now used in over 80% of Dutch schools. The principles underlying this pedagogy were strongly influenced by the notion of mathematics as a 'human activity' (Freudenthal, 1983). The Netherlands is considered one of the higher achieving countries in the world in mathematics (TIMSS, 1999, 2007; PISA, 2012) and consequently a number of other countries have attempted to design curricula based on the principles of RME.

RME in the USA – Maths in Context (MiC)

In 1991, The University of Wisconsin (UW), funded by National Science Foundation, USA and in collaboration with the Freudenthal Institute (FI), started to develop the MiC approach based on RME. The initial materials, drafted by staff from FI, were first published in 1996 after extensive trialling. The scheme has undergone several revisions since then, and is currently being modified in the light of the new US Core standards.

RME in the UK – Making Sense of Maths

In 2003, the Centre for Mathematics Education at MMU purchased a set of MiC materials and trialled them with Year 7 classes in a local school. This led to a three year trial of the materials in 12 Manchester schools. (See Dickinson and Eade (2005) for an account of the trials and findings, and Hanley and Darby (2007) for an account of research into the changes in teachers involved in the project.) The reaction to the materials was extremely positive, and there was a real sense that this approach was worthy of continued exploration

As a consequence of the MiC trial, a new project, *Making Sense of Maths* (MSM), was launched in 2007 aimed at KS4 pupils. This began with Foundation tier pupils and was then extended to include both tiers of the new two-tier GCSE. This project was in collaboration with the Freudenthal Institute in The Netherlands, working with Mathematics in Education and Industry (MEI) in the UK.

The focus of this paper is on the development of the MSM Higher level materials and how we have interpreted the original design principles of RME for these materials. The issues of context, models and progressive formalisation have already been explored in relation to Foundation level work (for example see Dickinson et al., 2009), but these created some very different challenges as we began to work at a higher mathematical level. We will provide one example where the design principles are relatively easy to see, and one where we initially struggled to adopt an RME approach. For a more general discussion of RME, and other design principles associated with it (for example, interactivity in the classroom), see Treffers (1991).

Theoretical Framework

Use of context

The use of context in mathematics teaching is not a new idea. Contexts are often used as a means of providing interesting introductions to topics, and then for testing whether or not pupils can use their knowledge to answer 'applications' questions. In RME, however, context is used not only as a means of applying previously learned mathematics but as a means of constructing new mathematics (Fosnot and Dolk, 2002). In this respect, context is seen as both the starting point and as the source for learning mathematics (Van den Heuvel-Panhuizen, 2003), and contexts are carefully chosen to encourage students to develop strategies and models that are helpful in the mathematising process. These contexts need to be experientially real to the students, so that they can engage in purposeful mathematical activity. At Foundation level, we regularly found that we could use real-life scenarios as our context, but at Higher level these were not always immediately available. In this situation, we often found ourselves looking at the historical development of the topic, and attempting to see where the need for the mathematics first arose. This not only provided us with a range of rich contexts but is also consistent with Fredenthal's notion of 'guided reinvention', which is central to RME (Freudenthal, 1983).

Use of Models

In RME, models bridge the gap between informal understanding connected to 'reality' on the one hand, and the understanding of more formal systems on the other (Van den Heuvel-Panhuizen, 2003). Crucially, a model that may initially simply be a model of a student's mathematical activity has the potential to develop into a model that will support and facilitate increasingly abstract mathematical reasoning. Many such models which emerged at Foundation level are immediately recognisable (eg. the double number line) although others would often be attributed a different label in the UK (e.g. repeated subtraction, ‘fitting in’). These models could sometimes be adapted for use at Higher level but, where this was not the case, it was often initially difficult to see what models would be appropriate in terms of the qualities discussed above.

Progressive formalisation

The term ‘progressive formalisation’ describes a learning sequence that begins with informal strategies and knowledge, and then develops into pre-formal methods that remain linked to concrete experiences, models and strategies. These models and strategies then develop progressively into more formal and abstract mathematical procedures (Webb and Meyer, 2007). This is strongly connected to the notion of ‘vertical mathematisation’ (Treffers,1991). Crucial to this process is a shift in how students view and use their models – from a ‘model of’ a situation (where the model is closely connected to the context) to a ‘model for’ (where the model is more abstract and can be used as the basis for mathematical reasoning) (Streefland, 1991).
An example of this, where the sharing of sandwiches evolves into a fraction bar and then a double number line, is shown below.

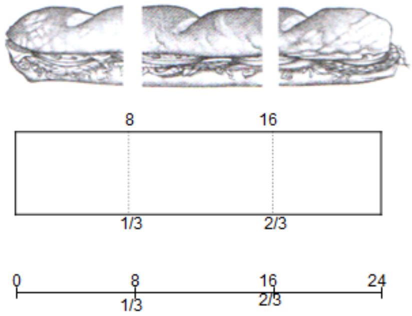


Figure 1. Progressive formalisation of models (Dickinson and Eade, 2005).

The issue of progressive formalisation is crucial, both to RME and to its adaption to the UK classroom. Clearly, a teacher wants students to be able to understand formal methods and procedures, nowhere is this more evident than at Higher level. RME does not shirk this responsibility, but offers a very different story of how students and teachers work towards this aim. For example, in relation to the teaching of fractions, the formal notion of equivalence has been categorised as being ‘on the horizon’ (Fosnot and Dolk, 2002) or the ‘tip of the iceberg’ (Webb, Boswinkel and Dekker, 2008).

Theory into practice

In some areas of the curriculum, Higher level topics are extensions of topics already met at Foundation level. Often, in these cases, the contexts and models used at Foundation level could also be extended.

So, for example, in a unit of work on Data Handling, the emergent model was a dot plot of some discrete data. The contexts used for this included estimating ten seconds, recording 100m Olympic finishing times and comparing long jump lengths. One example of the latter is shown below. This also provides a nice example of the ‘model of … model for’ interplay; in the diagram below, some students see the model as a representation of the ‘sand pit’ and have even been known to put stick drawings instead of dots (model of), while others now see this as a mathematical representation of the data (model for).

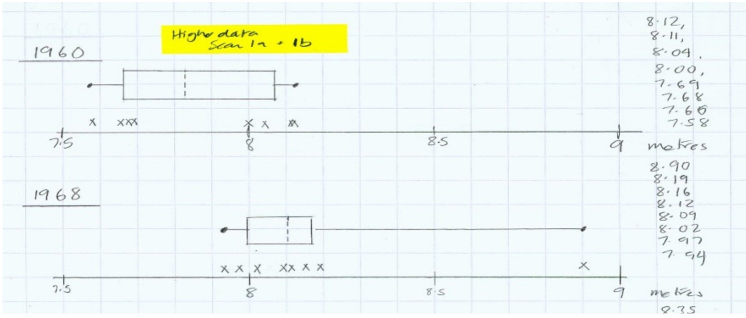


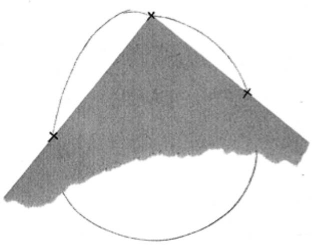
Figure 2. Using the dot plot as a ‘model of’ and as a ‘model for’.

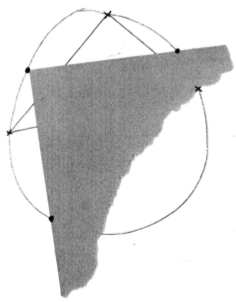
This model could now be extended to become a ‘box plot’, with consideration of measures at a higher level. Interestingly, within the context of comparing different people’s estimation of 10 seconds, it became apparent that ‘consistency’ was very important, and from this, students began to develop measures akin to mean deviation. So in this respect, the context and model not only help to develop the mathematics on the Higher level curriculum but can go further than that.

In other areas of the Higher level curriculum, however, it was initially more difficult to see a learning trajectory based on RME. For example, part of the Shape strand concerns ‘circle theorems’. No ‘context’ or ‘model’ was immediately obvious here, until we began to look at the historical development. From Thales’ theorem regarding the angle in a semi-circle, it became apparent that this was also used to ‘test’ for a perfect circle; from this developed the context of a circle drawing competition as shown below.

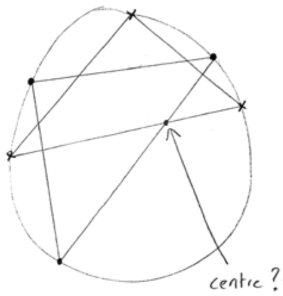
> Mrs Ayad the mathematics teacher is setting an end of term challenge. She has a prize for the person who can draw the best circle using only a pencil. As you can imagine, there was great excitement in the class! Mrs Ayad had an unusual way of judging the circles. She placed a piece of paper on a circle so that the 90° corner was on the circumference of the circle.

Figure 3. Testing for a perfect circle, adapted from Thale’s theorem.





She then did this again within the same circle



She said that where the two lines across the circle meet must be the centre of the circle

Through this the idea of the angle in a semi-circle emerges, but the formal maths comes from the mathematical activity of the students, and not the teacher!

Interestingly, using a fixed angle on a piece of paper can also be used to develop other circle theorems including ‘Angles in the same segment are equal’, and ‘Tangent and radius meet at 90°’. The fixed angle then becomes a model, in that it is bridging the gap between students’ informal ideas and the formal mathematics. In the classroom, some students wish to continue to use the piece of card whenever possible, some simply refer to it, while others begin to draw angles in the circle. These represent different stages of the journey from using a ‘model of’, to using a ‘model for’ and hence different stages of formalisation.

Research methodology

The most appropriate methodology for the purposes of developing RME based Higher level GCSE materials was to use ‘design study’ (Cohen, Manion and Morrison, 2011). In this sense we were designing a product that would be tested in real conditions, under observation and re-developed to take account of the findings. Fourteen teachers took part in trialling some of the six available modules, over a three-year period from 2010 to 2013. Five teachers had been project teachers in the earlier Mathematics in Context project; they had a strong sense of how students progress through the use of contexts and models with Foundation level topics. Five teachers had recently trained at MMU, four teachers were completely new to RME.

Teachers attended half-termly meetings after school hours where they were introduced to the materials and asked for feedback on the modules they had tried. Other forms of data collection included lesson observation and teacher interviews. Where possible lessons were video recorded and all interviews were audio recorded. This provided a large quantity of data which was later analysed in relation to the use of context, the use of models and a third emerging issue; that of teacher development.

Issues relating to the use of context

Experienced RME project teachers were able to recognise the importance of students’ being able to access the chosen contexts. For example, when a Ferris wheel is chosen as the starting point for a module on trigonometry, the students are first asked to imagine themselves on the Ferris wheel and to decide where on the ride their height would increase the most and the least. It emerged that students were finding it difficult to distinguish between vertical height gain and the circular direction of movement. Students were unable to estimate the vertical height gain as a distance because they did not know what distance this referred to in their diagrams. Rather than resorting to marking on the distance, the teacher went back to the context, video footage was shown and the students were asked to imagine dropping a stone out of their capsule; how would it travel, where would it land? This gave students an image of vertical height that they could now relate to.

On another occasion, the context of filling 3D Perspex objects with orange juice was used as part of a learning trajectory designed to enable students to access the relationships between formulae for the volume of a cube, a pyramid, a cone and a sphere. Through ‘fitting in’ the liquid, students could check their estimates for how many square-based pyramids it would require to fill a cube with the same sized base. The teacher with less experience in RME chose not to demonstrate this. His year 10 class were soon to do an end of year test and he felt under pressure to move quickly

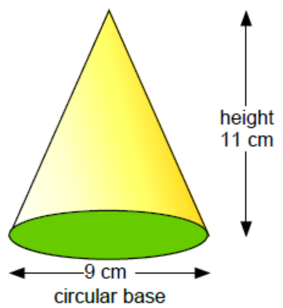


Figure 4. Formal representation of a cone

to the end of the module whereby students would engage with the traditional formal content of finding volume by substituting into the correct formulae.

We observed a student puzzling over finding the volume of this cone. When asked to describe what he saw in this picture, the student said he saw an oval shape and a triangle. He was unable to relate this representation to a real life object and would appear to be viewing this problem as one of area. Part of the sense making process in RME is to move slowly from the context (in this case the 3D Perspex cone) through informal representations of the context (student's own hand drawn 2D representations) to the more formal representation shown above (Figure 4). It would seem at Higher level, teachers inexperienced in the use of RME, driven by the need to cover formal content, at times chose to dilute this aspect.

Issues relating to the use of models

One of the issues at Higher level is that the sense making features of a particular context or model may not apply to the degree of abstraction required. For example, although the bar model works well for numerical fractions, it becomes extremely challenging to represent the addition of algebraic fractions on a bar model. The area model for multiplication, where numbers are contextualised as lengths, is much harder to conceive for algebraic expressions of the form $(x - 2)(2x - 6)$, where the use of negative numbers would imply cutting back into line lengths already drawn. Our teachers dealt with these issues in a number of ways. In schools where classes had previous experience of working with RME models at KS3, teachers were able to build on the previously established context to model experiences and extend these to algebraic thinking. For example, students who were able to conceive the addition of $\frac{1}{3}$ and $\frac{1}{5}$ as drawing a bar which would partition into both 3 and 5 parts, when asked how many pieces would be required to add $\frac{1}{a}$ and $\frac{1}{b}$ referred to needing 'a lots of b parts'.

In another case, a project teacher was so delighted with the way the context of buying hats and T-shirts had prompted sense-making models for solving simultaneous equations, that she persisted with questions relating to the context even when the equations were contextually difficult to interpret. Students faced with solving $2x - 3y = 12$ and $2x + 7y = -8$ were asked, 'Where are you getting more?' 'How much more?' and 'How much more does it cost you?' These questions acted as prompts to use the models students had previously developed.

Where classes had little experience of RME, some teachers understandably felt unable to invest time exploring situations where the models emerged from the context. Instead, they introduced their students to models as methods in their own right. For example, students were taught to expand 2 brackets by drawing a grid. The grid gave no attention to the relative 'size' of the various terms and when the material required students to describe where in the picture the 'x^2' could be seen, it was clear that they were thinking not in terms of area, but in terms of the procedure used to generate the 'x^2' term.

Teacher development

Teacher interviews conducted at the end of one or two years' engagement with the material revealed a number of common threads. Teachers were extremely positive about the use of context, not only as a motivational tool, but also as a memorable point of reference long after the module was completed. In revision sessions, teachers described themselves and their students as using the language of context to evoke imagery and associated methods, in addition to their more usual practice of revisiting routines.

The role of discussion featured heavily in the teachers' appraisal of what had changed. One teacher new to the RME approach described how discussion had enabled him to 'stop driving some elements of the lesson … you can allow people to have time to find things out for themselves, to discuss what they're doing.' He goes on to acknowledge his changing teacher identity: 'It's a lot less about me driving them through something, explaining something for them to regurgitate …. I just kind of perch and find interesting things that are going on, then periodically stop people and say, "right, let's have a look at what you're doing". This teacher would appear to be beginning to develop the complex teacher role of framing student contributions.

By contrast, an experienced RME teacher also commented on the rich opportunities for discussion but acknowledged the skill required by the teacher to 'steer them in certain directions.' She would appear to be aware of the importance of the teacher who is operating in an RME frame pro-actively steering student ideas and representations towards more formal thinking. (Gravemeijer et al, 1998)

A conflict of interest in the current educational climate

Although teachers viewed progressive formalisation and the increased status of discussion as positive features, they also expressed concern about how these would be viewed by the current monitoring bodies. Schools' interpretation of Ofsted guidelines and the National Numeracy Strategy (1999) has led to teachers being encouraged to demonstrate progress for individual students within the course of one lesson, to itemise objectives at the start of a lesson, and to ensure a lesson has 'pace'. Teachers felt there was a disparity between these requirements and RME based lessons where students were given time to express opinions, to share a range of solution strategies and to slowly make progress towards formal thinking. In addition, the then current practice of entering students for modular GCSE, plus repeated early entry, even for Higher level students, meant a lot of the KS4 curriculum time was spent preparing for tests and examinations. This on occasion resulted in teachers withdrawing from particular modules, and is an issue that we continue to work on. The increased focus on rigour in the new 2014 National Curriculum is likely to increase the pressure on teachers to deliver formal methods quickly, to emphasise acquiring the procedure and so create a further source of potential conflict with an RME based approach.

Conclusion

In conclusion, we believe that it is possible to develop materials for Higher attaining students that fit within the design principles of RME. To do this, however, requires an extension to how we might traditionally interpret the notions of context, models, and progressive formalisation. Some contexts and models extend from Foundation to Higher level, but this is not always possible. We found that looking at the historical development of topics often suggested a context and model, as in the work on circle theorems. In other areas, previous learned mathematics may become the context for new work. In terms of formalisation, it is as important as ever that students are allowed to work informally and 'make sense' of situations before developing more formal mathematics. On the other hand, teachers at Higher level report feeling under pressure to move to formal mathematics quickly, and as curriculum designers, we clearly need to reflect this in the materials we produce. Our work at both Foundation and Higher level suggests that, for many teachers, working with RME can be a challenging (though ultimately very rewarding) experience. Although the materials are designed to support the teacher in the classroom, Professional Development is essential if the materials are to be used effectively. This is particularly true where teachers feel under pressure from examinations and other external factors. We believe that an RME based curriculum is suitable in the current UK climate, but that teachers need training and direct classroom support for this to be truly effective. The nature of this CPD, and how it can be successfully managed, is an area for more study.

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