Possible parallels between visual representations and informal knowledge

Leonardo Barichello

University of Nottingham

This paper is based on a case from the pilot of my PhD research project with a group a secondary students. I will argue that visual representations can work as a basis for reasoning about addition of fractions for low achieving students, similarly to what was shown by Nancy Mack regarding informal knowledge for multiplication of fractions.

Keywords: visual representation; low achieving students; informal knowledge; fractions

Pursuing meaning and reasoning for fraction operations

Julia (pseudonym) has been a mathematics teacher for 12 years in a secondary school. Year after year, she teaches fraction operations to all her groups. Her approach is mostly based on the procedures for each operation and even includes mnemonics, such as ‘cross and smile’ for addition and subtraction and ‘keep, flip and change’ for division to help her students remember the procedures.

According to her, the approach usually works in the short term: most of the students get good marks in the next exam. However, every year, especially with the lower sets, she has to teach the procedures again as they have never seen them before. Anecdotally, I had the opportunity to observe students in her Year 10 Set 1 unable to calculate $\frac{3}{8}-\frac{2}{8}$ or applying the full ‘cross and smile’ procedure instead of $\frac{3-2}{8}=\frac{1}{8}$.

The goal of this paper is to present the first steps towards an approach emphasizing visual representation to teach addition and subtraction of fractions that would allow students to make sense of these operations through diagrammatic reasoning (Rivera, 2011). However, before discussing the data from my pilot study, I will present below the work developed by Nancy Mack, whose goal was very similar to mine, but who used informal knowledge instead of visual representations.

The case of informal knowledge

Mack (1990) is the first of a series of papers in which the author investigated the possibility of using students’ informal knowledge as a basis to build understanding for fraction operations. Her motivation arose from research indicating that this is the case for whole number arithmetic and, therefore, could also be the case for fractions.

At first, the author focused on fraction addition and subtraction (Mack, 1990; 1995). After conducting individual instructional sessions with 15 students from Years 4, 5 and 6 of average mathematical ability, the data highlighted that “students' informal knowledge was initially disconnected from their knowledge of fraction symbols and procedures” (Mack, 1990, p. 16) and also that “knowledge of rote procedures frequently interfered with students’ attempts to build on their informal knowledge,” (Mack, 1990, p. 16). Although Mack emphasizes that these studies reinforced the proposal that understanding should come prior to rote knowledge, I would argue that her conclusions suggest that the use of informal knowledge as a basis for understanding is not a simple endeavour when it comes to the addition and subtraction of fractions.

Later, the author shifted focus to multiplication of fractions (Mack, 2001). The methodology applied was similar to the previous studies and:

> The results show that students' informal knowledge of partitioning did not fully reflect the conceptual complexities underlying multiplication of fractions; however, all six students were able to build on this knowledge to reconceptualize and partition units in a variety of ways. Thus, the results of this study illustrate that it is possible for students to build on their informal knowledge to develop an understanding of a complex mathematical content domain (Mack, 2001, p. 292)

Differently from the previous studies, this one showed that students were able to solve a variety of problems involving fraction multiplication by relying on the informal idea of 'partitioning into equal parts', reinforcing the possibility of using informal knowledge as a basis for understanding fraction multiplication instead of exclusively depending upon teaching procedures.

The case of visual representations

Mack was not particularly interested in visual representations in her studies, however a quick look at the papers reveals that the students often utilized this sort of resource to solve the questions proposed by her.

In this paper, visual representations are any external representation (meaning available publicly, as in a piece of paper, instead of internal, only available mentally to the person) in which the topological arrangement and geometrical properties of the elements are important and there is no clear 'order of reading'. It may be seen as the opposite of a textual representation, which is sequential and symbolic, meaning that the symbols used have no similarity with the 'things' they represent.

Although there is a gradient from visual to textual representation, the characteristics of the extreme cases may be useful to clarify the advantages and disadvantages of each.

Textual representations	**Visual representations**
Symbolic	Iconic
Based on natural language	Based on topological disposition of elements and their geometrical properties in a 2D or 3D space
Manifested through speaking and writing	Manifested through some kind of drawing
Captured through viewing and hearing	Captured through viewing
Essentially sequential	The disposition of the elements are not sequential, but carries meaning in terms of the relationship between them.

Although "language is a marvellous tool for communication, [...] it is greatly overrated as a tool for thought" (Reed, 2013, p. 1). Among the advantages of visual representations, there are two that are particularly relevant to this study.

The first refers to arguments showing that visual representations may facilitate certain kinds of inferences. Larkin and Simon (1987) showed this effect by comparing the solution of a typical pulley problem by purely textual means to a solution using a diagram. It not only helped to search, register and organize information, but also facilitated some inferences.

The second advantage arises from the perception that visual representations may be particularly beneficial for low achieving students. In a meta-analysis of his own studies, Mayer (1997) points out that:

> students who lack prior knowledge will be less likely than high prior knowledge learners to independently create useful mental images solely from the verbal materials. Thus, low prior knowledge learners are more likely than high prior knowledge learners to benefit from the contiguous presentation of verbal and visual explanations (Mayer, 1997, p. 15)

Gates (2015) further discusses this issue and identifies other possible explanations for the phenomenon, such as the reduction of the dependence on writing and verbal skills, which are highly correlated to low achievement in general.

Altogether, these advantages motivated the approach adopted in the pilot study, which emphasizes the use of visual representations, and will be further explored in my data collection.

A pilot study about visual representations and fractions

The context

Before the beginning of the 2014-2015 academic year, my supervisor contacted the Head of the Mathematics Department of a secondary school in Nottingham willing to establish a commitment to a long term collaboration focused on students with low achievement levels in mathematics.

At that moment, the school had been placed under special measures by Ofsted (indeed, most of the secondary schools in the city were similarly graded on the basis of the percentage of pupils gaining 5 or more A*-C grades at GCSE). The majority of the students are of white British origin and around one fifth are from minority ethnic groups. The percentage of students in receipt of free school meals (43%) is significantly higher than the national average (28%). The head of the Mathematics Department perceives the school as ethnically homogeneous in comparison to other schools in which he has worked before. According to him, this is due to the predominantly working class catchment area. In terms of attainment, he stated that the school receives students slightly below the national average and that the students also perform slightly below national average at GCSE. They are placed in sets according to prior attainment in mathematics from Year 7 to Year 11.

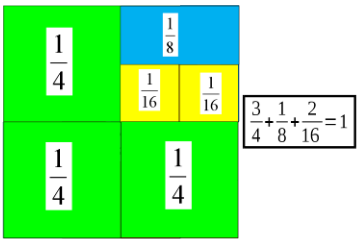
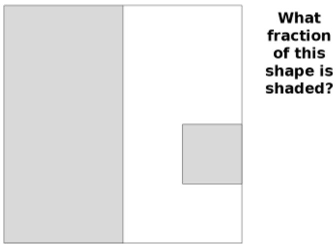
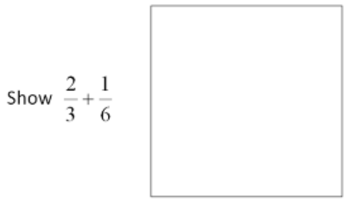
Lesson observations and meetings

In November 2014, I joined my supervisor and another PhD student in observing lessons for low sets in Year 7, 8 and 9 from three different teachers. During the next 5 months, I observed about 30 lessons and interacted with teachers both before and after the lessons in order to get used to and, hopefully, understand the British educational context, the school and their daily practices.

In February 2015, we started a series of meetings between the three researchers and three teachers to design lessons about fractions (topic chosen by the teachers based on their perception of what would be beneficial for Year 7, 8 and 9 students in the lower sets) emphasizing visual representations.

The first meetings were mostly composed of discussions regarding tasks and lesson plans coherent with our goal. The material varied hugely from lesson plans from MAP Project (http://map.mathshell.org) and ICCAMS (http://iccams-maths.org/) to tasks from Youcubed (https://www.youcubed.org/). At this point, the goal of the research team was to bring a host of different ideas to the teachers and evaluate how they would react to them.

After three meetings, it was agreed that one of the teachers, Julia, would work on lesson plans covering fraction addition for her Year 7 Set 4 (out of 5). From this point on, all the meetings focused on discussing her lesson plans. By the end of the academic year, the group developed three lesson plans and Julia enacted all of them. The research team was present at all of these lessons taking field notes and video-recording them. The table below shows an overview of the tasks and topics covered during each lesson.

Lesson 1	Lesson 2	Lesson 3
- Definition of fraction using the rectangular area model; - Build a whole using 1/2, 1/4, 1/8 and 1/16;  -Identify equivalent fractions.	- Identify sums with the fractions from the previous lesson shown on a diagram and compute it;  - Represent in a diagram and solve a sum given symbolically.	- Introduce 1/3, 1/6 and 1/9 using the same model; - Match equivalent fractions given in diagrams and symbolically; - Represent and solve a given sum with the new fractions.  $\frac{2}{3}+\frac{1}{6}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$

For the first and second lessons, the students had coloured cut-outs representing all the fractions that they could use to solve the tasks. The third lesson was based solely on diagrams drawn by the students. The lessons were spread out from 15th of May to 17th of July and, between the lessons, Julia systematically posed one question about fractions (similar to the tasks proposed in the last lesson) as a starter. These were the only lessons about fractions this group of students had in the whole year.

Evidences of reasoning

One focus of the research team was to look for signs of appropriation of the visual representations by the students, similar to that defined by Rivera (2011) as diagrammatic reasoning.

Among many pieces of evidence showing that students were actively using visual representations, such as drawings on their worksheets, explicit verbal references to the cut-outs when explaining their solutions and gestures when solving the questions, one solution to a question posed in the end of the third lesson is particularly interesting.

The question was simply “Show $\frac{1}{9}+\frac{5}{6}$”.

This question was not included in the lesson plan and was posed to one student, Graham (a pseudonym), because he had already completed all the tasks. It is important to highlight that the students only worked out sums involving fractions whose denominators were multiples of each other. Therefore, this question was beyond what was explicitly discussed in the lessons and the research team had no idea how the student would approach it. Figure 1 shows Graham’s solution.

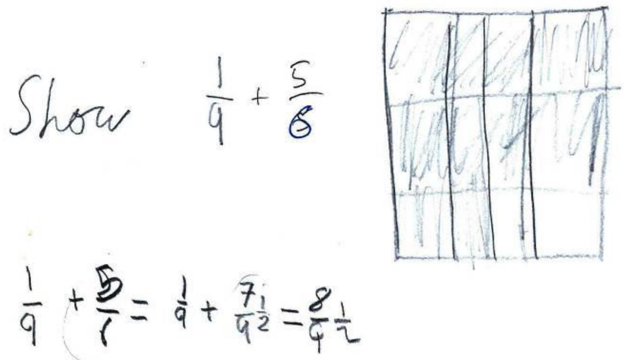


Figure 1: Graham's solution.

His final answer is far from conventional and at no point in the lessons was this sort of answer presented or suggested by the teacher. However, it seems reasonable to interpret it as below and, therefore, he is correct.

$$\frac{8\frac{1}{2}}{9}=\frac{8.5}{9}=\frac{17}{18}$$

The most impressive feature of his solution is how he used the diagram to obtain the final answer. The re-construction below was based on what was observed by one of the teachers attending the lesson and by careful analysis of the solution above. Note that the writing in the bottom of his solution suggests that he used the diagram to find a fraction equivalent to $\frac{5}{6}$ with a denominator equal to 9.

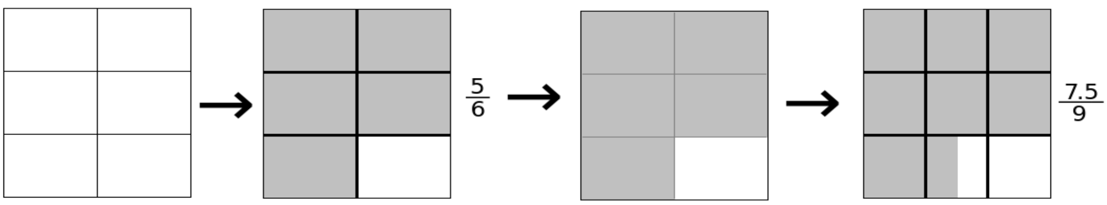


Figure 2: Re-construction of the diagram in Graham's solution

From this interpretation, it is reasonable to say that Graham could not have solved the question without the support of the diagram and, at the same time, that the diagram allowed Graham to move beyond what was explicitly taught to him in such a way that suggests understanding of equivalent fractions and of the meaning of addition of fractions.

This observation resonates with the idea of diagrammatic reasoning proposed by Rivera (2011). According to the author, once students understand the representation and the rules behind it, they may use it as tools for reasoning and not only as ways of representing ‘things’. This is a step further than reading, writing and transforming one representation into the other.

Moreover, the fact that the representation used was visual may have enhanced its potential as a tool for reasoning, as shown by Larkin and Simon (1987). In the solution presented above, I would argue that the representation allowed the realization, by visual means, that $\frac{1}{6}$ is the same as $\frac{1.5}{9}$, which was of the utmost importance for his solution.

Discussion

The studies conducted by Mack (1990, 1995, 2001) pointed out that informal knowledge can be used as basis for developing an understanding of fraction multiplication for students of average ability in mathematics and also identified some limitations that seem to compromise the possibility of using it for fraction addition and subtraction. The importance of this kind of conclusion is that it sheds light on ways of achieving conceptual understanding for a topic that is recognized as challenging by mathematics teachers.

Although only one case was presented in this paper, the other pieces of evidence observed during my pilot study and the arguments in the literature favourable to the use of visual representations in the teaching of mathematics suggest that this approach is promising in terms of promoting the understanding of fraction addition and subtraction, particularly for low achieving students.

References

Gates, P. (2015). Social Class and the Visual in Mathematics. In S. Mukhopadhyay and B. Greer (eds) *Proceedings of the 8th International Mathematics Education and Society Conference*, Portland State University, Oregon, Vol 2, pp. 517-530

Larkin, J., & Simon, H. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science*, 99, 65–99.

Mack, N. K. (1990). Learning Fractions with Understanding: Building on Informal Knowledge. *Journal for Research in Mathematics Education*, 21(1), 16–32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building an informal knowledge. *Journal for Research in Mathematics Education*, 26(5), 422–441.

Mack, N. K. (2001). Building on Informal Knowledge through Instruction in a Complex Content Domain: Partitioning, Units, and Understanding Multiplication of Fractions. *Journal for Research in Mathematics Education*, 32(3), pp. 267–295.

Mayer, R. E. (1997). Multimedia learning: Are we asking the right questions? *Educational Psychologist*, 32(1), 1–19.

Reed, S. K. (2013). *Thinking visually*. Psychology Press.

Rivera, F. D. (2011). *Toward a visually-oriented school mathematics curriculum: Research, theory, practice, and issues*. Springer Science & Business Media.