Applied mathematics = Modeling > Problem solving?

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The latest stage of an ongoing investigation of the case for reforming Level-3 mathematics, with modeling replacing traditional applied mathematics, is reported. The case rests on the potential for improvement in learners' ability to use mathematics knowledge and improvement in take-up. Issues around teaching model making are identified and discussed and also the importance of mathematical modeling in understanding knowledge and progress in science.

Introduction

The topic covered by this paper is part of an ongoing investigation by the author of the Level-3 mathematics curriculum and its utility (Osmon 2009). Progress overall is sketched in below to provide a perspective on the current topic. I have been focussing on the consequences for universities of the relatively low take-up of Level-3 mathematics (essentially confined to the post-16 sub-cohort with A-in-GCSE-mathematics) at a time of HE expansion, and the following summarises what has become apparent. I have looked at the amount of Level-3 mathematics needed to undertake undergraduate courses across a whole range of subjects at universities of differing academic standing, and found this "foundational mathematics" is characteristic of the subject and largely independent of the academic standing of a university (although more prestigious universities certainly demand higher grades). I have been able to separate subjects into four groups, according to their foundational mathematics needs: A: Traditional STEM subjects, B: Computer Sciences, C: Finance and Management oriented subjects, D: Business, Health, and Social Sciences. I have called Groups A, B, and C the Quantitative Subjects and Group D the Semi-quantitative Subjects. Subjects across both Groups B and C need the equivalent of one year of foundational mathematics at Level-3, but while group C subjects need a close approximation to the pure mathematics content of the traditional AS, Group B (Computer Sciences) needs a year of discrete mathematics- with very little overlap with the pure mathematics content of the traditional curriculum. (The content of the traditional curriculum is prescribed nationally as 2/3 pure mathematics and 1/3 applied. The pure portion is also prescribed in detail, but choice of content is available for the applied- principally units in mechanics or statistics, but more recently decision-mathematics and financial-mathematics have been specified.) It is remarkable that the curriculum does not provide for the modest foundational mathematics need of the large numbers of students entering courses in Group D subjects: they need to learn some statistics- one or two units- alongside their main Level-3 subjects. Perhaps there is actually a good case for students entering courses in any of the quantitative subjects to have this knowledge. With compulsory post-16 mathematics on the education reform agenda, this is worth consideration.

Prior to collecting UCAS data for my subject groups, I informally interviewed academics responsible for quantitative courses at several of the universities in my sample and a frequent comment was that often the students they admitted were unable to use their mathematics knowledge outside the narrow confines of textbook exercises and short examination questions, despite the allocation of 1/3rd of the curriculum to applied mathematics. Informal interviews with tutors on those HE courses admitting students with only Level-2 mathematics identified these students' lack of mathematical activity, during the two and a half years interval between taking GCSE and beginning their undergraduate course, as a concern. They suggested this inhibited acquisition of the needed foundational mathematics in "top-up" classes in their first undergraduate year that aimed to get them up to speed mathematically. So, besides the issue of poor take-up of mathematics at Level-3 there is an issue of maintenance of their Level-2 mathematics knowledge, for the many students who will need to use, and perhaps enhance, their modest mathematics knowledge in their university courses. So, the shortcomings of the Level-3 mathematics curriculum are not confined to the issue of a relatively small take-up: effectively just the A-in-GCSE-mathematics post-16 sub-cohort. There are also issues concerning curriculum content: coverage of discrete pure mathematics, coverage of how to apply mathematical knowledge, and keeping Level-2 mathematics knowledge alive, and even enhanced, for students not specialising in mathematics.

Modeling may supplant traditional applied mathematics

I am a self-taught modeller, with experience of using and making models in Physics, Electronics, Computer Science, and Management and I have assumed (for at least two reasons) that modeling is not a subject capable of being taught in the classroom- instead it can only be learned, as I learned, by doing it. Although I can envisage an experienced modeller mentoring apprentice modellers, I am sceptical that many mathematics teachers would have the necessary knowledge and experience. Besides, the model making situations I found myself in were very diverse and, except in the most broad brush sense, no general method of attack seemed to be applicable- in contrast with the very procedural nature of most school mathematics.

Recent publications seem to imply I am unduly pessimistic. The AQA Use of Mathematics AS-level qualification (AQA 2009) includes a significant modeling component and Margaret Brown (Brown 2011) draws attention to its success in opening a door into Level-3 mathematics for the B/C sub-cohort. Real-world Problems for Secondary School Mathematics Students (Maas and O'Donoghue 2011) is a resource book for teaching modeling, comprising case studies contributed by enthusiastic European mathematics educators.

Does my modeling experience provide me with any insights to contribute to the development of modeling teaching? Up to now, in my study of the Level-3 curriculum I have not considered the applied mathematics content. In the traditional curriculum, applications- applied mathematics- are separated from pure mathematics, and presented as a small set of disjoint topics: mechanics, financial mathematics, decision mathematics, statistics, that tend to be characterised by small, closed problems and associated prescriptions for solving them. How might this change if learning about mathematics applications through modeling becomes the norm?

Models: what they are and what they are for

At this point clarification of terms seems desirable. Firstly, it is easier to give some examples showing the diversity of models than an all encompassing definition: reduced scale wooden models of building developments; mechanical models of animal joints; visualisations such as video simulations of planetary motion in the solar system; business models, such as- "pile them high and sell them cheap" for a market trader. Secondly, it is important to be clear about the *purpose* of models: they are an aid to *comprehension* of real-world artefacts and phenomena. NB The role of a model, even a mathematical model, is to give insight- not numbers!

Mathematical models are a special class of model. Mathematics describes pattern/regularity. Mathematics cannot describe the real-world with all its irregularities but it can describe an idealised version. Hence wherever pattern or regularity exists, in a real-world phenomenon or artefact, there is the possibility of description of these aspects in mathematical language: a mathematical model. For example, the structural part (at least) of a family tree is a mathematical model of family history. Mathematical models are important because they can be so *powerful*. They are not merely descriptions. They can have *predictive power* too. Some examples are: the orbits of planets and their moons and the prediction of eclipses; weather forecasts; reliability of computer systems; causes and consequences of global warming; the spread of diseases, etc, etc. Mathematical models are often expressed as sets of equations and this is particularly apparent in the sciences. Scientific knowledge is largely a collection of more or less connected mathematical models. These models encapsulate our knowledge of real-world phenomena and help us to comprehend it- by reference to related *ideal-world* ("model") phenomena. Our understanding of complex real-world phenomena is likely to be woolly beyond the scope of the model- where the i*deal-world* ceases to be an adequate representation (model) of the *real-world.*

Using models

Historically, most professions developed a collection of models expressed as heuristics- "rules of thumb". An example from the author's youth, important to all potential house buyers, was one's mortgage-ability: "a 30-year mortgage is affordable if it is no more than 3 times you salary". Nowadays these simple models have been supplanted by more elaborate software versions packaged as computer applications, and most professional domains have them. For example: Project management; Actuarial calculations; CAD-CAM; Just-in-time stock control; Patient monitoring; Traffic control. When the user's role is simply to supply the application with parameter values, professional knowledge has been transferred to the software model maker with consequent de-skilling of professional roles.

Many remarkable software models are accessible over the Internet to mathematics students and their teachers. Typically these describe population growth and decay, or spread of diseases, under various conditions selected by the user. When the outcomes of these, perhaps complex, dynamic processes are communicated using visualisation software and typically brightly coloured displays, then the effect can be dramatic and (hopefully) much more insightful than mere numbers. However, while interaction with such models may motivate one to study modeling, the experience probably does not actually teach much mathematics.

Making a model

In contrast with using an existing model, the author's experience is that actually making a model does develop one's mathematics. It would be nice to have a generally applicable procedure for making models: a series of steps for the model-maker to take in order to progress from a real-world problem situation to a model. The author has not encountered such a procedure, although afterwards, in a particular case, when the job is done it may be possible to write out the key development steps along the way. There are always two major steps in getting from the real-world problem domain (Rw) to the model M: *abstraction* and *idealisation*. Abstraction is about simplifying the problem domain: important stuff is transferred (abstracted) from Rw into a simpler Abstracted-world (Aw) leaving unwanted complications behind. For example I may decide to ignore air resistance to a falling body. Idealisation simplifies the problem domain still further because Mathematics only deals with patterns/regularity. For example real lines and shapes in the Aw are transformed into geometric lines and shapes in this Idealised-world (Iw). (The nature of the Idealisation depends on the kind of mathematics used- in this case geometry.)

An example is probably helpful at this point. Consider the DIY problem of Tiling a Bathroom Floor: a model is needed to determine how many tiles are needed and where to place them (aesthetically). Abstraction- the walls of the room are somewhat irregular, so in order to simplify focus attention on the main area of floor, away from the edges. A (geometric) grid of squares *idealises* this main area. Counting the squares solves the first part of the problem and positioning the grid (translation/rotation) by trial and error for best aesthetic effect, solves the second part. Evidently size of tile and orientation of the grid are user-determined parameters. Of course the regions around the walls have been omitted from the model and they need tiling too, and it may be that "best aesthetic effect" in the main area is not the optimal arrangement if the appearance of the edge regions is taken into account, then the grid location may be adjusted somewhat (including these details *refines* the model). The example is very straightforward but often there are many false starts and the outcome may not be a very satisfying model. Making a model is not a matter of following a general procedure: one gets better at it with practice. *But is it a teachable skill?*

Idealisation depends on knowing some appropriate mathematics. The story goes that when Einstein was developing his theory of general relativity he realised he needed non-Euclidian geometry- which he didn't know- but he knew a man who did. At Level-3, students know only a very limited range of mathematics and this constrains their model making. The curriculum includes very little discrete mathematics which is quite limiting considering the role of information technology in our society. With a modest knowledge of set theory students would be able make relational data-base models and even investigate data mining. However the curriculum does include some probability and probabilistic modeling can be fun.

A historic example of scientific model development

Arguably, the history of science is the story of such model development processes. Sometimes these models get refined over many years, sometimes they prove entirely inadequate to account for some recently observed real-world phenomenon and a fresh start is necessary. The kinetic theory of gases survived several such refinement stages. The following is a brief overview.

Real-world observations: In experiments with various gases the approximate relationship P.V/T = constant was observed between pressure P, volume V, and temperature T.

Model: A gas comprises *small* and *perfectly elastic spheres* in random motion at a speed u dependent on T, and P is caused by the spheres impacting the container walls.

Abstractions: small (eliminates complication of collisions among spheres) uniform speed u (eliminates complication of speed distributions)

Idealisations: small (allows V to tend to zero at large P), perfectly elastic (energy is conserved in collisions with enclosing walls)

Refined Model: needed for agreement with more accurate experimental data and further insights into the structure of matter: Spheres (molecules) have a finite size; Collisions are not perfectly elastic (short range forces between molecules); Speed distribution among molecules.

Cross-curricula issues

Scientific knowledge is largely a collection of more or less connected mathematical models. This immediately raises a cross-curricula issue between mathematics and science, to which presumably mathematics should make a proportionate contribution. But how? One possibility is explored in the COMPASS project (COMPASS 2009) where parallel but separate threads are explored concurrently in science and mathematics classes. Model making is also related to designing and this poses another cross-curricula issue, this time between Mathematics and Design-Technology. I claim, but space does not allow for elaboration, that the basic principle of a *Design* is very like a first-stage Model. A *Design* in a particular real-world context requires extension from this so as to optimise performance, or reliability, or cost, or for user convenience, and this is very like the process of refining a Model.

Insights and Eureka moments

Making an original model is not a smooth process. Rational analysis of the problem domain only takes one so far. What follows, typically, is seemingly endless worrying at the puzzle. In my experience, a sudden jump of understanding delivers the final insight to end the process, accompanied, in due course, by a great sense of accomplishment, a Eureka moment in fact. Part of the satisfaction seems to come from being able to state the insight very succinctly, almost like a slogan.

World-changing Eureka moments:

Archimedes- a floating body displaces its own weight model of water; Darwin-evolution by natural selection; Dirac- two solutions to his equation of quantum electrodynamics implying the existence of antimatter as well as matter.

Some of my personal insights are:

Stools with 3 legs won't wobble; when the mortgage rate is lower than the rate of inflation I am being paid to borrow money (if I can get a mortgage); insure only against the hits I can't afford to take; failure of integrated circuit chips is closely analogous to radio-active decay of isotopes (I can use the same mathematics- Poisson statistics); computer architecture is about trade-offs among performance/reliability/ cost/usability (this became the theme for a course of lectures); a point on a page printed with a grid of squares is either on a grid-line/on a point-of-intersection/within a square (this insight led to a patent application).

Eureka moments for students doing modeling:

My Eureka moments were exciting and have given me lasting pleasurable memories. If every student of modeling could have a Eureka moment it would surely do wonders for the take-up of Level-3 mathematics.

Conclusions

It is evident from the literature that many mathematics educators, in Europe and the US as well as here, are enthusiastic about the potential that doing modeling has both for motivating mathematics learning and for enhancing learners' mathematical knowledge. The author's personal experience is that making a model is an absorbing, infuriating, and ultimately highly rewarding activity and we should surely give Level-3 learners this experience if we possibly can. In the UK, where we have a particular problem of low take-up of Level-3 mathematics, the success of the AQA Use of Mathematics AS-level qualification, with its emphasis on modeling, has demonstrated the potential value of modeling for widening access. However, qualified mathematics teachers are a scarce resource. In most schools, at least in the state sector, it may be unrealistic to expect concurrent Use of Mathematics and Traditional Mathematics streams to exist side-by-side. Perhaps pump-priming funding, until sufficient take-up of Use of Mathematics gets established, would be a way forward.

Learners following the traditional A-level mathematics stream would surely benefit if its applied mathematics units were converted to model-making units. And there is also the issue, identified in the Introduction, of keeping Level-2 mathematics knowledge alive, and even enhanced, in the case of students not specialising in mathematics at Level-3. With compulsory post-16 mathematics under discussion, perhaps a proposal for all post-16 students to undertake some statistical modeling, alongside their Level-3 major subjects, would be timely.

However, teaching modeling will challenge teachers to adjust to unfamiliar ways of working. There are issues around methodology, project work, and teachers' knowledge. While all mathematics is learned by doing, it is not apparent that there is a general teachable method for building models, in contrast with mathematics generally. And making a model takes longer than solving a traditional applied mathematics problem and so almost inevitably is done as project work, and perhaps in teams. Further, mathematics teachers inevitably lack the breadth of domain knowledge they would need to be authoritative mentors across a range of modeling problems so that they must adjust to an unfamiliar more arms-length role as project supervisor rather than instructor. But mathematics teachers' response to the challenge of technology-PC labs turning mathematics into a laboratory subject- gives reason for optimism.

The discussion, above, of modeling's cross-curricula links, including the place of mathematical models in science presents yet further reform curriculum issues.

References

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