Exploring algebraic thinking in post-16 mathematics: the interpretation of letters.

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Students in their first year of post-16 mathematics were given a test consisting of items requiring algebraic reasoning. This was based on work by Kuchemann with secondary school students. The responses were analysed to assess students' level of algebraic thinking and their results compared with their public examination results. This paper includes a summary of the analysis and a discussion of the implications.

Keywords: algebra; post-16

Background and motivation

Algebra, as many would argue, is key to the development of mathematical skills (for example, MP Kelvin Hopkins' question in the House of Commons (Hansard 2011); Professor Smith's report into post-14 mathematics education (2004, 86)). Indeed, much of the Core 1 module (C1), taken in the first year of post-16 mathematics in England, involves algebra, including manipulating quadratic functions, solving simultaneous equations, summing series and finding equations of straight lines in multiple forms. It is also an area that I have found C1 students often have difficulties with. This gave me the motivation to investigate the issues that students face. The theoretical framework was taken from Küchemann's work where he had examined secondary school children's' understanding of numerical variables (Küchemann 1978; Hart 1981). For my dissertation for an MA in Mathematics Education I decided to revisit Küchemann's study, applying it to students who had just sat the C1 examination and analysing their written responses using, as an analytical model, the categories that Küchemann had identified.

Methodology

Küchemann's framework combined the interpretation of letters with structural complexity to relate to Piaget's levels of concrete and formal thinking. To identify what level of formal, or abstract, thinking students required and the level of abstraction that was required to achieve high grades, I formulated the following research questions:

- What level of abstract algebraic thinking is necessary for success at the Core 1 module?
- To what extent is abstract algebraic thinking being tested?

As my study was largely based on Küchemann's approach I will first give an outline of his methodology.

Küchemann's research was carried out as part of the Concepts in Secondary Mathematics and Science project between 1974 - 9. He took the position that algebra in secondary school is 'generalised arithmetic' (Hart 1981, 102), which I interpreted as 'the use of letters for numbers and the writing of general statements representing given arithmetical rules and operations' (Booth 1984, 1). He used two preliminary criteria for deciding on the content of the diagnostic tests and for assessing the results. The first was the 'structural complexity' of the test items, for instance, the number of variables. The second criterion was the interpretation of letter that was required to solve the question. Küchemann classified six different interpretations of letter, summarised in Table 1.

Category	Interpretation	Example question and commentary
Letter evaluated	The letter is assigned a numerical value from the outset.	Find '*a*' in $a + 5 = 8$. No manipulation of letters is required.
Letter not used	The children ignore the letter, or at best acknowledge its existence but without giving it a meaning.	Given $a + b = 43$ find $a + b + 2$. One method is to match off the common letters and the only operation is to add two numbers.
Letter used as an object	The letter is regarded as shorthand for an object or as an object in its own right.	The length of each side of an equilateral triangle is given as *e*, find the perimeter. By reducing the abstract object to something concrete, as a label for an object for instance, the difficulty of the problem is significantly reduced.
Letter used as a specific unknown	Children regard a letter as a specific but unknown number, and can operate upon it directly.	If $e + f = 8$, give an expression for $e + f + g$. While '$e + f$' can be matched off as 8 the answer still requires manipulation with an unknown.
Letter used as a generalised number	The letter is seen as representing, or at least as being able to take, several values rather than just one.	What can you say about *c* if $c + d = 10$ and *c* is less than *d*? The aim here is to see if the students will give several values for c, rather than perceiving c as a specific number to be found.
Letter used as a variable	The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.	Which is larger, $2n$ or $n + 2$? Explain. One way to approach this is to look at how $2n$ and $n + 2$ each change as n changes; and then compare the rates of change. Thus the method requires building 'first-level' relationships and then comparing them, hence forming a second order relationship.

Table 1. Definitions of interpretation of letters (Hart 1981, 104).

Test items were sorted into four groups. Levels 1 and 2 consisted of the first three categories of interpretation, levels 3 and 4 comprised items where the letter needed to be interpreted as a specific unknown, general unknown or variable. The structural complexity helped inform whether the item should be placed in the higher or lower of the two levels in each case. The allocation to levels was refined in the light of the empirical results; some items were problematic and did not necessarily sit neatly in one or other of the levels. Küchemann suggested a correspondence between these levels and the Piagetian stages of cognitive development as shown in Table 2.

Level 1	Below late concrete
Level 2	Late-concrete
Level 3	Early-formal
Level 4	Late-formal

Table 2. Correspondence between levels of algebraic thinking and Piagetian stages of cognitive development.

Based on the examples in Küchemann (1978) and Hart (1981) I compiled a test with 7 items at level 3 and 9 at level 4. The test was given to our students who had recently sat the C1 examination; 162 scripts were returned, 145 with names which I could duly collate with their C1 results.

Results

Levels of algebraic thinking and C1 scores

The relative difficulty of questions remained constant compared to Küchemann's results (see Table 3) except for four items. Items 4ii and 9 were solved much more

Item Number	Algebra Thinking Test (Core 1 students) Facility (% correct) (n = 162)	CSMS Test (15 year olds) Facility (% correct)	Level in Hart (1981)	Level in Mathematics in School (1978)
4ii	94	25	4	4
5i	89	56	3	3
9	89	16	4	4
6	88	35	3	3
5ii	88	-	-	-
3	85	41	3	3
4i	85	41	3	3
1	77	39	3	3
8	77	30	4	3
5iii	77	32	4	4
2	69	50	3	3
12	68	10	4	4
13	68	16	4	4
11	65	8	4	4
10	60	13	4	4
7	59	27	4	3

Table 3. Comparison of the facilities of test items and their levels.

successfully; these both involved the expansion of brackets. In contrast, item 2 ("If $f + g = 8$, what does $f + g + h$ equal?") was classified as level 3 but was answered with the same success as level 4 items. Item 7 ("Consider the statement $L + M + N = L + P + N$. Is it true always, sometimes, or never?") was classified as level 3 in Küchemann's first study and then at level 4 in 1981; it proved to be one of the hardest items for the C1 students.

Following Küchemann's approach, to 'pass' a level two-thirds of the questions had to be answered correctly at that level. While the levels were considered to be hierarchical, 5 students failed level 3 but passed level 4. Excluding these exceptions, the results identified 28% of students (n=140) working at level 3 or below (see Table 4).

	Failed level 4	Passed level 4
Failed level 3	14	5
Passed level 3	25	101

Table 4. Numbers of students and their achievement at level 3 and level 4 in the algebraic thinking test.

A breakdown of level of algebraic thinking against unclassified (grade U), low (grades C to E) and high (grades A and B) C1 grades is shown in Table 5. Using the chi squared test with this data (n = 140, excluding the 5 exceptions) with the null hypothesis that the level of algebraic thinking has no bearing on the C1 grade gained (U, C to E, A to B) gave a statistic of 17.680 which is significant at the p=0.01 level with 4 degrees of freedom (13.277). This suggests there was some association between C1 grading and the level achieved on the algebraic thinking test.

	C1 grade U	C1 grade C to E	C1 grade A - B	
< Level 3	2	8	4	14
Level 3	4	19	2	25
Level 4	13	36	52	101
	19	63	58	140

Table 5. Two way chart showing level of algebra thinking against C1 grading.

While a high percentage of those who achieved high C1 grades also achieved level 4 in the algebra test, attaining level 4 did not guarantee a high grade. Moreover, failing to achieve above level 3 did not preclude gaining an A or B grade at C1. Interestingly, the median C1 score for those failing to achieve level 3 was considerably higher than those who achieved level 3, as shown in Figures 1 and 2.

Figure 1. Interquartile range for C1 marks for students below level 3 (n = 14).

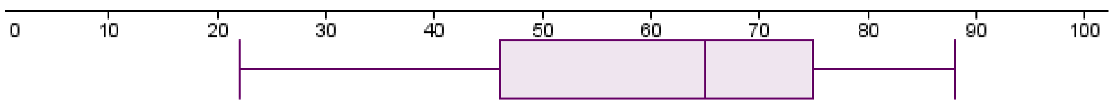


Figure 4. Interquartile range for C1 marks for students achieving level 3 (n = 25).

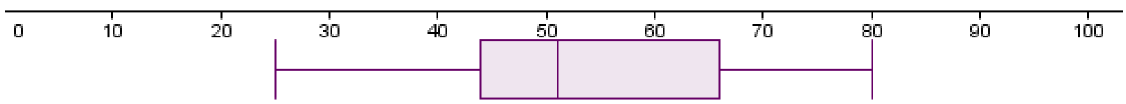
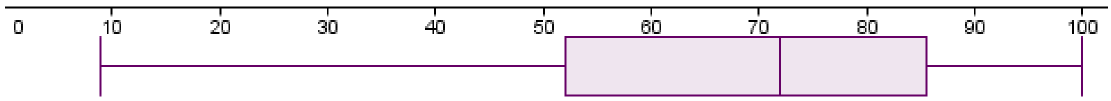


Figure 3. Interquartile range for C1 marks for students achieving level 4 (n = 101).



The interpretation of letters

In addition to the above quantitative analysis, the written answers were examined to understand what interpretations students were giving to the letters. In summary the following tendencies were noted:

1. Letters were thought of as ‘objects’ rather than standing for numbers.
2. Letters were assigned numerical values from the outset, rather than being seen as unknowns to be manipulated.
3. A letter represented a specific number; different letters had to represent different numbers.
4. A letter was seen as a specific unknown rather than a generalised number.
5. A letter was perceived as standing for just a few possible values, perhaps restricted to discrete positive values, possibly extending to decimals, fractions or/and negative numbers. Sometimes there was evidence of the student recognising further possibilities as they thought through each set of results and considered the implications.
6. Interpretation depended on the perceived context, e.g. a formula as against an equation.
7. Letters were associated with particular roles; often x and y were introduced if the student felt two unknowns were needed.
8. More than one interpretation might be used in a single question.

There was also evidence that some students conceived of a letter as representing a unique object at the same time as being considered as a generalised number, suggesting that ‘letter as object’ can exist at various levels of formal thought, not just in concrete thinking.

Discussion and conclusions

Comparison of the facilities of the diagnostic items suggests that the majority were of a similar relative difficulty. The main exceptions were those involving the expansion of brackets which appeared to be a routine operation for many, and empirically these items fell into level 3.

A concern was raised regarding the conditions under which the algebra test was administered. While the C1 examination was carried out under rigorous conditions, the algebra test was given in the classroom and there was some evidence that there had been some collaboration in two of the classes amongst some students. However, while this may account for the extended range at the lower end of the C1 range for level 4 (see Figure 3), this was on a small enough scale not to undermine the results.

From the assessment of the written results the ability to think formally emerged as a factor in performance at C1. However, while the use of late formal thinking was an advantage in achieving higher C1 grades, it was neither a necessity nor a guarantor. This was seen in the wide spread of C1 marks across the cohort for level 4 students (given there may have been some collaboration), while some students working at level 3, or below, achieved A or B grades. Thus some students still achieved good grades without necessarily working in the abstract. Indeed, it was demonstrated that it was possible to achieve an A grade without working at level 3, suggesting that some students appeared to have alternative strategies for dealing with the more abstract questions.

The analysis of students' interpretations showed that restrictions on the meanings they accessed constrained the solutions available to them. Interpreting the letters as objects was not unusual and, where they were interpreted as generalised number, quite often the possibilities were restricted to a few positive integers. The perceived context influenced the interpretations used while in some responses it was evident that students used more than one interpretation as they thought through the problem. These results suggest that there are opportunities to develop methods and resources to aid thinking with the more abstract interpretations as well as to foster flexibility in shifting between meanings.

Acknowledgements

This work was carried out as part of my Master's degree in Mathematics Education, which was partly funded by Havant Sixth Form College.

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