# BSRLM Geometry working group: the role of the teacher in teaching proof and proving in geometry

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It remains the case the the geometry component of the school mathematics curriculum is viewed as providing key opportunities for teachers to develop learners' capacity for deductive reasoning and proving (as well as their spatial and visualisation capabilities). Nevertheless international research consistently shows that teaching the key ideas of proof and proving to all students is not an easy task. Through a consideration of an example of classroom teaching, this paper scrutinises a selection of theoretical frameworks that may afford greater insight into, and greater explanatory power on, the role of the teacher in the teaching of proof and proving in geometry. The brief sketches provided of three relevant theoretical frameworks (the theory of socio-mathematical norms, the theory of teaching with variation, and the theory of instructional exchanges) illustrate the fact that there is much scope for further suitably-designed research studies.

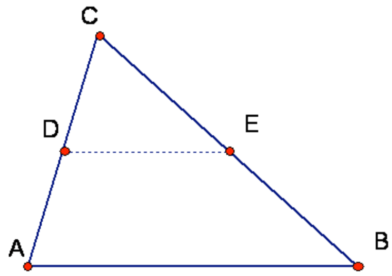
**Keywords: theory, teaching, geometry, proof, proving**

## Introduction

The geometry component of the school mathematics curriculum, in most countries, provides not only an opportunity to build learners' spatial and visualisation capabilities, but it is also a key vehicle for developing their capacity for deductive reasoning and proving (Battista, 2007; Fujita & Jones, 2007; Jones & Rodd, 2001; Royal Society, 2001). Yet international research consistently shows that teaching the key ideas of proof and proving to all students is not an easy task (for recent reviews, see, for example, Mariotti, 2007; Mariotti and Balacheff, 2008). What is more, as Stylianou, Blanton, & Knuth (2009: 5-6) identity in terms of existing research, "very few studies have focused on the teaching of proof in the context of teachers' day-to-day instructional practice". In recognising that more is currently known about the *learning* of proof than about its *teaching*, this paper focuses on the role of the teacher in teaching proof and proving in geometry. The structure of the paper is to begin with an exemplar case of classroom teaching and examine this through the lens of a selection of theoretical frameworks. The intention is to see in what ways these theoretical frameworks may afford (and perhaps constrain) greater insight into, and greater explanatory power on, the role of the teacher in the teaching of proof and proving in geometry.

## An example of classroom teaching in geometry

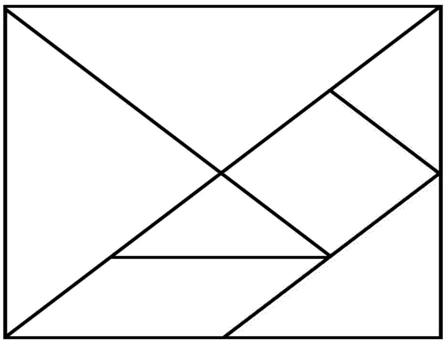
The example of classroom teaching selected for this paper is aimed at students at lower secondary school (upper middle school). The teaching builds through a series of classroom tasks (in part, derived from tasks from *Connected Geometry*, produced by the Education Development Center, 2000) that lead students to proving the theorem illustrated in Figure 1 (known as the mid-point, or mid-line, theorem).




*Triangle ABC has D and E as midpoints of sides AC and AB respectively.*
*Is DE parallel to AB? If so, why?*

Figure 1: Classroom task based on the mid-point (or mid-line) theorem

The first classroom task in the teaching sequence is based around the Tangram puzzle, a set of seven shapes that form a square (see Figure 2). The task is to use all seven pieces to make a rectangle that is not a square.



*Use all seven pieces to make a rectangle that is not a square.*

Figure 2: Classroom task based on the seven-piece Tangram

The next classroom task in the teaching sequence involves creating a specific quadrilateral out of a several Tangram pieces and, by reflecting, rotating, or translating one or two of the pieces, transforming the given quadrilateral into another quadrilateral. For example, a parallelogram can be formed out of three Tangram pieces (a square and two triangles) and the problem is to transform this into a rectangle (the solution is to translate one of the triangles).

The next classroom task is the reverse – here the task is to begin with a specific quadrilateral and find a way to dissect it (i.e. cut it up into Tangram pieces) and rearrange the pieces to form another specific quadrilateral. For example, dissect a parallelogram into pieces that can be rearranged to form a rectangle; or start with a right-angled triangle and dissect it so that the pieces can be rearranged to form a rectangle.

The next classroom task entails looking in more detail at the problem of dissecting a right-angled triangle so that the pieces can be rearranged to form a rectangle. As illustrated in Figure 3, by rotating the small top triangle 180° about the midpoint of the hypotenuse of the right-angled triangle, one obtains a rectangle.


Figure 3: dissecting a right-angled triangle to form a rectangle

In Figure 3, the two segments of the hypotenuse (of the right-angled triangle) match because the cut was at the midpoint of the hypotenuse.

The following are the conjectures that can be used in a proof of the mid-point (or mid-line) theorem: quadrilateral ABCD is a parallelogram because the opposite sides are the same length (in Figure 3, AD and BC are the same length because they were made by cutting at a midpoint); AB and CD (in Figure 3) are the same length because a mid-point (or mid-line) cut makes a segment half as long as the base.

From this, as illustrated in Figure 4, it is possible to move to a proof of the mid-point (or mid-line) theorem.

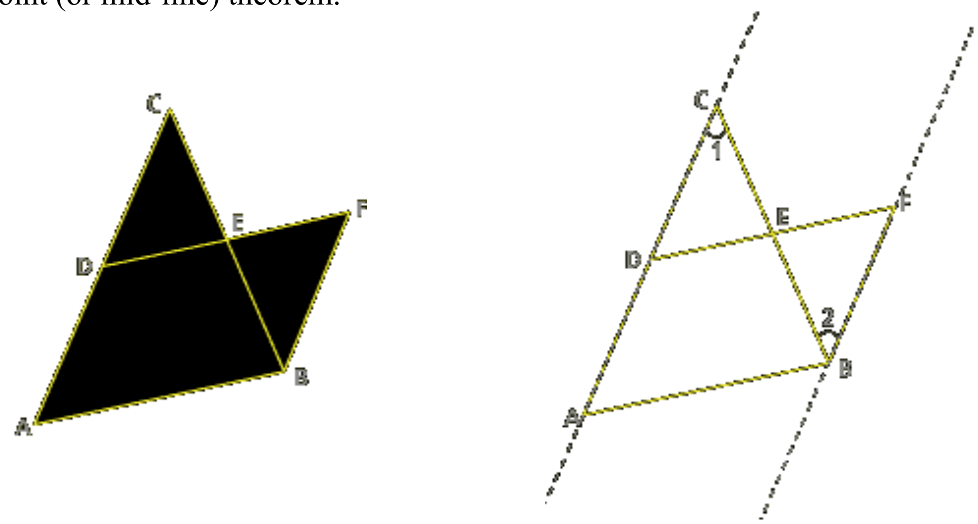



Figure 4: proving the mid-point (or mid-line) theorem

Relevant steps in the proof include explaining why triangles DEC and FEB are congruent, why AD and BF have the same length, why AD and BF are parallel, why ABFD is a parallelogram, and why DE is parallel to AB and half as long.

Extension tasks are then possible in which the mid-point (or mid-line) theorem can be used to solve other geometrical problems.

No doubt a series of lessons based on the above flow of tasks might be taught badly. What is more, it is not possible to capture within this short paper all the features of the effective teaching of actual classroom lessons based on the tasks. Hence, for the purposes of the analysis below, the assumption is that the teaching fitted with what is known about effective teaching in general (Hay McBer, 2000; Muijs & Reynolds, 2001); i.e. that the lessons displayed high expectations of the students, that the planning was clear and well-thought-out, that a suitable variety of teaching strategies and techniques were used to engage the students and keep them on task, that a clear strategy for pupil management was in place, that time and resources were managed wisely, and that a range of assessment techniques were employed to monitor student understanding and inform ongoing teaching.

## Theoretical frameworks of the role of the teacher

Given that a multitude of theories exist in mathematics education that, in one way or another, are concerned with proof and proving, Jones and Herbst (forthcoming) argue that it is worth focusing on the following three: the *theory of socio-mathematical norms*, the *theory of teaching with variation*, and the *theory of instructional exchanges*. In what follows, each of these three theoretical frameworks is briefly brought to bear on the above taught sequence of lessons.

### *The theory of socio-mathematical norms*

The notion of *socio-mathematical norms* (Voigt, 1995; Yackel and Cobb, 1996) provides a framework for analysing how normative understandings are promoted by the teacher in their role as classroom representative of the discipline of mathematics. An example of such a norm is what counts as an appropriate mathematical justification within the classroom setting. In the above taught sequence of lessons, the teacher would be involved in promoting such norms across each classroom task – from the first task (in explaining why the shape made by all seven pieces is a rectangle that is not a square) to the final task (of explaining, in Figure 4, why triangles DEC and FEB are congruent, why AD and BF have the same length, why AD and BF are parallel, and so on).

In proposing this notion of socio-mathematical norms, Yackel and Cobb (1996) provide a means to understand how the notion of a proof as *an explanation accepted by a community at a given time* could both result from the interaction and negotiation among individuals in response to the values and practices of the discipline of mathematics exemplified through the acts of the teacher in handling the student contributions to each lesson. In a suitable research project, micro-analysis of classroom interactions could be used to track the development of shared norms about the mathematical practice of proving.

### *The theory of teaching with variation*

From the perspective of the *theory of teaching with variation* (Gu, Huang, & Marton, 2004), classroom tasks are aimed at ensuring students gain experience of two types of variation - "conceptual variation" (providing students with multiple experiences from different perspectives) and "procedural variation" (providing students with experiences of the process of forming concepts). In the above taught sequence of lessons, the teacher could be said to have incorporated both forms of variation – by varying the Tangram puzzles in particular ways, and by providing students with varying experiences of providing mathematical explanations that lead to a suitable proof for the theorem that underlies the sequence of classroom tasks.

In this way, the theory of teaching with variation may be useful in analysing the teaching of proof and proving in mathematics. In a suitably-designed research project, analysis of teachers' use of variation could be used to track the development of the concept of proving and the procedures of doing proofs.

### *The theory of instructional exchanges*

The theory of instructional exchanges (Herbst, 2006; Herbst, Miyakawa & Chazan, in revision; Jones & Herbst, forthcoming) is based on the view that classroom teaching proceeds as a sequence of exchanges (or transactions) between, on the one hand, the moment-to-moment (and possibly interactive) work that students do with their teacher and, on the other hand, what the teacher views as having been accomplished mathematically. Thus, a form of 'contract' exists that makes the teacher responsible for attending not only to the student as learner of mathematics but also to mathematics as a discipline. Not only that, but the work of the teacher includes managing tasks across two different timescales: the work done moment-to-moment (at the scale of the utterance) and the work at the larger scales of the month-long or even year-long curriculum stages that are associated with establishing mathematical objects of knowledge.

In the above taught sequence of lessons, the teacher could be said to be attending to both the student-teacher 'contract' (in terms of not only attending to student learning but also to the mathematics) *and* managing classroom tasks so that the students are aware of specific tasks (for example, a dissection task) *within* the overall framework of working towards a proof of the mid-point (or mid-line) theorem. In a research project, an analysis of the teachers' role in instructional exchanges could focus on how this develops student understanding of, and competence with, proof and proving.

## Concluding comments

This paper began by noting that “very few studies have focused on the teaching of proof in the context of teachers’ day-to-day instructional practice” (Stylianou, Blanton, & Knuth, 2009: 5-6). This underlines the fact that there is much scope for suitably-designed research studies. This paper has provided brief sketches of three possible theoretical frameworks might inform the design of such studies - the theory of socio-mathematical norms, the theory of teaching with variation, and the theory of instructional exchanges.

Within the scope of this short paper these three theoretical frameworks might be seen as competing – in other words, as a possible example of what Sfard (2002) calls the over-proliferation of theorising. As such, it might thence be worth considering how further research might meet the challenges set forth by Sfard not only to increase efforts to carry out research studies within existing theoretical frameworks, such as the ones covered in this paper (with the intention to establish the range of applicability or validity or usefulness of the theories), but also to carry out comparative surveys of theories that purport to provide frameworks for dealing with the same topics and to compare the terminologies used by different theories in order to identify cases where different terms are used for essentially the same idea or where the same term is used to designate ideas that are essentially different. While such studies are needed at lower secondary school (upper middle school) level, there remains much scope for appropriate research at the primary (elementary) school level (Sinclair and Jones, 2009) as well as at the upper secondary school and tertiary level.

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**BSRLM Geometry Working Group**

The BSRLM geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. The aim of the group is to share perspectives on a range of research questions that could become the basis for further collaborative work. Suggestions of topics for discussion are always welcome. The group is open to all.