Introducing the concept of infinite sum: Preliminary analyses of curriculum content

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In this paper we report the first phase of a study that aims to analyse curriculum content, pedagogical practice and student perceptions of the complex, often counter-intuitive but significant mathematical concept of infinite sum (aka series in Calculus). Sources of student difficulty with the concept identified in previous, not very extensive, research include: certain student perceptions of infinity; limited exposure to visualisation, contextualisation and applications of infinite sums; and, teaching through reduction to an algorithmic approach. Here we report preliminary analyses regarding curriculum content and, in particular, the initial phases of a three-dimensional analysis (cognitive, epistemological, didactical) of mainstream texts used to introduce the concept to undergraduates in the UK.

Keywords: Infinite sum, visualization, application, university mathematics

Learning and Teaching the Concept of Infinite Sum

The work we report in this paper is the first, self-contained, phase of a study that investigates the learning and teaching of a complex, often counter-intuitive but significant mathematical concept, the concept of *infinite sum* (aka as *series* in Calculus). The applications of infinite sums in mathematics and science are wide ranging and crucial (e.g. González-Martín & Nardi 2007). In mathematics, for example, infinite sums are a fundamental element of the process of calculating the area under a curve, a calculation with a vast array of applications in Economics and Physics. In Medicine and Biology infinite sums provide ways of modelling situations such as the distribution of medications or pollutants. In sum infinite sums are central to the mathematical education of a wide range of scientists and professionals. It is therefore quite surprising that the studies of its learning and teaching are rather few.

Students' difficulties with the concept of infinite sum have been reported mostly indirectly in the works that study the concept of convergence (e.g. Robert 1982) – often in the context of the infinite sums underlying some mathematical situations such as integration (e.g. Fay & Webster 1985). These studies suggest that misunderstandings of the concept of infinite sum may originate in perceptions of infinity, such as that the sum of infinitely many quantities is always infinitely great, and may result in some of the difficulties with understanding the concept of Riemann integral and, particularly, improper integral (e.g. González-Martín 2006). These studies also suggest that the absence of visual understanding (e.g. Alcock & Simpson 2004) associated to the concept of infinite sum poses severe limitations in students' understanding and application of the concept (e.g. Mamona 1990).

In sum students appear to have little understanding of what the concept actually means, have no visual imagery associated with it and see little or no relevance to it in mathematical and other situations. As is often the case with teaching of complex mathematical topics at upper-secondary and tertiary levels (e.g. Artigue et al. 2007), teaching, through reduction to an algorithmic approach (e.g. exercises that require an often blind application of formulae; static use of graphical representations; absence of a connection to other crucial concepts; no attempt to alter related misconceptions about infinity etc.) evades addressing students' difficulties.

An Analysis of Curriculum Content on Infinite Sums

The work we report here is a self-contained part of a larger international study currently in progress in the UK (first two authors) and Canada (third author). The larger study aims to investigate the learning and teaching of infinite sums through:

I. Study of the student learning experience with regard to:
   a. Analysis of curriculum content and pedagogical practice
   b. Analysis of students' perceptions

II. Design, implementation and evaluation of a pedagogical intervention that addresses student needs as emerging from I.

Here we report on the preliminary analyses regarding *Ia* and in particular on the UK chapter of above international study (we also summarise Canadian analyses).

Our work towards *Ia* consists of an analysis of mainstream texts used to introduce the concept to upper secondary and university level students. These texts include: books, lecture notes, exercise sheets, A level materials where the concept is introduced informally for the first time, etc.. Analogously to the three dimensions described by Artigue (1992) our analysis of the texts aims to address the following questions:

- *Epistemological*: what are the mathematical ideas these texts aim to convey, particularly in the light of the concept's history?
- *Cognitive*: what issues related to student learning do these texts aim to address?
- *Didactical*: what type of teaching are these texts conducive to, particularly considering the disciplinary and institutional context in which the concept is taught?

At this preliminary phase we have already identified, with the help of lecturers teaching the topic, seven of the texts that we will analyse, both from applied and pure mathematics, typically used by students following the introduction of the concept in the lectures. We will supplement this analysis with some observation of lectures and a small number of interviews with lecturers teaching the concept to undergraduates in mathematics, science and engineering. Here however we draw exclusively on a first-level analysis of the seven texts and our conversations with one lecturer. Our analysis addresses questions that have emerged from the literature and a preliminary analysis (González-Martín 2008) of six recent post-secondary texts.

These questions include:

- Does the text support – and how – students' overcoming of the major misconception tantalising the learning of the concept of infinite sum, 'infinitely many addends, infinitely great sum' through reference to examples from this concept's epistemology and history?
- Does the text use – and how – graphical representations in order to enrich students' visual understanding of the concept?

- In what order does the concept appear in the text (for example, in relation to the appearance of the notion of numerical sequence of which it is a logical precedent). And does this order – and how – take into consideration the fundamental differences between a mathematically 'appropriate' order and the ways in which students acquire a new concept?
- Does the text instil – and how – an algorithmic and mechanical approach to the concept (despite recent research and policy advice to the contrary)?
- Does the text contextualise – and how – the concept in terms of its *raison-d'être* in mathematics and its many applications in mathematical and other situations?

The initial impressions from González-Martín's preliminary analysis (ibid) with regard to the above questions are that, even though the concept enjoys substantial coverage in most texts, its presentation is largely a-historical and decontextualised, almost exclusively in the algebraic register and with few graphical representations and applications. The UK team (first two authors) is currently working closely with the Canadian team to substantiate and refine, or refute, these impressions.

Visualisation, Application and History in UK Texts

At this preliminary phase we have identified seven books proposed by lecturers teaching the topic to students of mathematics and other disciplines. These books are – amongst other texts – part of the mainstream material that is used for the introduction of the concept of series in applied and pure mathematics university and foundation courses in the UK (Bostock & Chandler 2000; Gilbert & Jordan 2002; Haggarty 1989; Kreyszig 2006; Priestley 2003; Spivak 1967; Stephenson 1973). Here we draw exclusively on a first-level analysis of these seven books and our conversations with one lecturer.

In accordance with the questions listed above we have recorded in a spreadsheet the following information on each book:

- The number of pages dedicated to the concept of infinite series.
- The number and type of figural representations (e.g. graphs, drawings etc.) and the ratio of representation per page.
- The number and type of applications of the concept of series (e.g. real life applications, applications in other disciplines, problem solving, modelling etc.) and the ratio of application per page.
- The number and type of historical references (e.g. a simple reference to events, integration of history in teaching etc.) and the ratio of references per page.

Regarding figural representations, only in three of the books we found figures related to series: three in Kreyszig (0.23 figures per page); five in Spivak (0.19 figures per page); and three in Stephenson (0.12 figures per page). These figures are used mainly for the visual representation of the sum terms or the partial sums as: points on the number line (Figures 1, 2 and 3) or areas of rectangles (Figures 4 and 5).

In particular, Figure 1 features a neighbourhood of $s(x_1)$, a visual expression for the inequality $\left|s\left(x_1\right)-s_n\left(x_1\right)\right|<\varepsilon$. Figure 2 features the partial sums s_1, s_2, s_3, s_4 of the series $x_1 - x_2 + x_3 - x_4 + \ldots$, where $\{x_n\}$ is a monotonic decreasing to zero sequence. According to the Leibniz Theorem this series converges and the illustration of Figure 2 supports the claim of the proof that:

$$s_2 \leq s_4 \leq s_6 \leq \ldots,\ \ s_1 \geq s_3 \geq s_5 \geq \ldots \text{ and } \ s_k \leq s_l, \text{if } k \text{ is even and } l \text{ is odd}$$

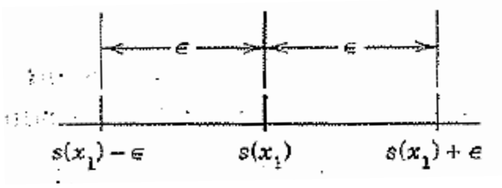

Figure1. Kreyszig 2006, 172

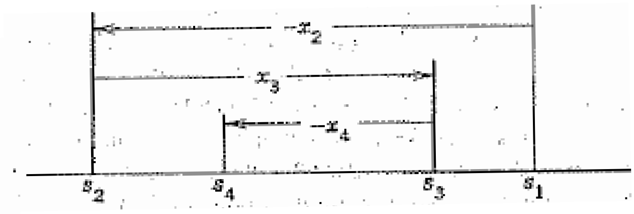

Figure 2. Kreyszig 2006, A70

Figure 3 features the terms and the partial sums of the series: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=1$. Through this picture, not only the order of the terms is illustrated but the convergence of the series is evident and as Spivak suggests this is "an infinite sum which can always be remembered from the picture" (Spivak 1967, 391).

Figure 3. Spivak, p.391

Figure 4 visualises the symbolic expression: $f(n+1)<\int_n^{n+1} f < f(n)$ for monotonic functions, whereas Figure 5 features "[…] a graphical argument. Each term of the series represents the area of the rectangle with base equal to the unity and height equal to the magnitude of the term" (Stephenson 1973, 72)

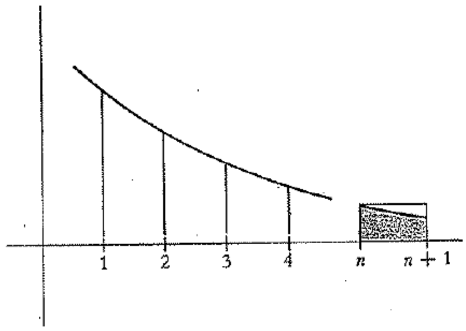

Figure 4. Spivak 1967, 396

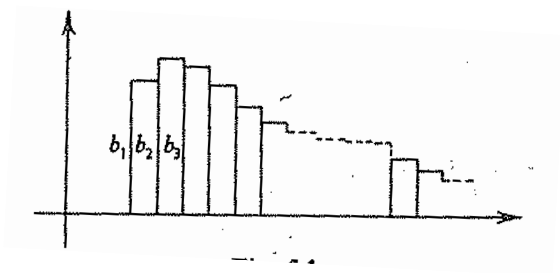

Figure 5. Stephenson 1973, 72

In some of the books we found the following non-figural but rather evocative representation of the proof of the non convergence of the harmonic series (via grouping of the terms):

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{9}+\ldots+\frac{1}{16}}_{\geq\frac{1}{2}}+\ldots$$

In a conversation we had with one of the lecturers teaching the topic, he proposed the figural representation of this grouping we present in Figure 6.

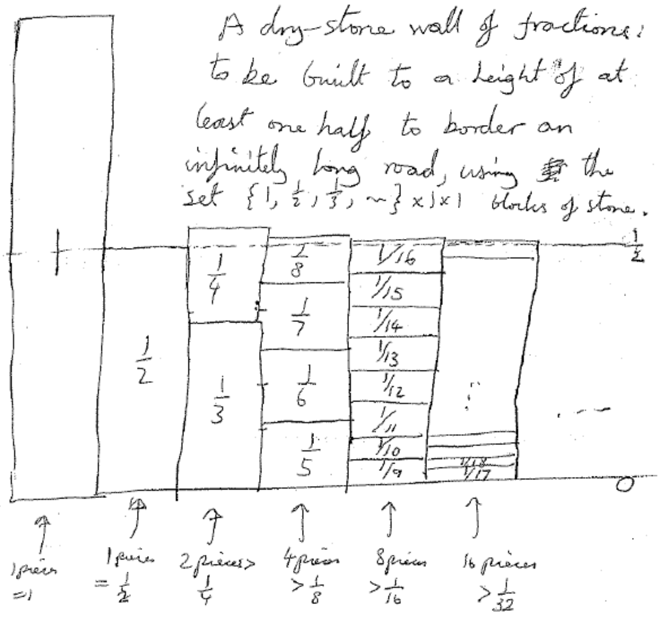

Figure 6. Lecturer's figure

Regarding applications, we found three in two books: two in Bostock & Chandler (0.10 per page) and one in Gilbert & Jordan (0.13 per page). In one of them, which only provided a context for a materially-based calculation, there was a problem of a piece of string with length *l* that was cut in half and kept one piece; the remainder in half and kept one piece; and so on in order to end up with the series described in Figure 3. In the other two applications only the formulas from other disciplines (economics and physics) are used in order the mathematical processes be applied. We note that in Spivak's book, although we found no non-mathematical applications, we identified a tendency for intra-mathematical connections, namely connections not necessarily between different disciplines but between mathematical topics. So, for example, the non-convergence of the harmonic series is connected to the discussion in an exercise on p.411 of the infinite number of positive rational numbers.

Regarding historical references, no historical reference was found in the seven books we studied for the purpose of this preliminary analysis.

Further Steps

As our analyses grow the list of research questions initially produced by the Canadian team is further enriched. For example, we are now looking more closely at intra-mathematical connections within the texts such as the ones we exemplified here with reference to the Spivak's text. The depth and breadth of these analyses will also be inevitably influenced by the outcome of our applications for funding (currently pending in the UK and already successful in Canada). This work is also embedded in the work of an international network that aims to consolidate the work of several researchers from around the globe on the teaching and learning of Analysis concepts, particularly with regard to student understanding of *real numbers*.

References

Alcock, L.J., and A.P. Simpson. 2004. Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics* 57 (1):1-32.

Artigue, M. 1992. Didactic Engineering. In *Research in Didactics of Mathematics*, ed R. Douady and A. Mercier, 41-65. Grenoble, France: La Pensée Sauvage.

Artigue, M., C. Batanero, and P. Kent. 2007. Mathematics Thinking and Learning at Post-secondary Level. In *The Second Handbook of Research on Mathematics Teaching and Learning*, ed F. K. Lester, 1011-1049. US: NCTM.

Bostock, L., and S. Chandler. 2000. *Core maths for advanced level*. 3 ed. Cheltenham, UK: Nelson Thornes Ltd.

Fay, T., and P. Webster. 1985. Infinite series and improper integrals: a dual approach. *Mathematics and computer education* 19 (3):185-191.

Gilbert, J., and C. Jordan. 2002. *Guide to mathematical methods*. 2 ed. New York: Palgrave Macmillan.

González-Martín, A. S. 2006. La generalización de la integral definida desde las perspectivas numérica, gráfica y simbólica utilizando entornos informáticos. Problemas de enseñanza y de aprendizaje. Doctoral Thesis, University of La Laguna.

———. 2008. The concept of infinite sum: a first review of textbooks, in Proceedings of the. In *Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education, and the XX North American Chapter*, edited by O. Figueras and A. Sepúlveda, vol. 1, 262. Morelia, Michoacán, México: PME.

González-Martín, A. S., and E. Nardi. 2007. A first approach to the teaching of series. Some results in Canada and UK. In *59ème Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM59)*. Dobogókô, Hungary.

Haggarty, R. 1989. *Fundamentals of Mathematical Analysis*. Wokingham, UK: Addison-Wesley Publishing Company.

Kreyszig, E. 2006. *Advanced engineering mathematics*. 9 ed. Hoboken, NJ: John Wiley & Sons, Inc.

Mamona, J. 1990. Sequences and series –Sequences and functions: Students' confusions. *International Journal of Mathematical Education in Science and Technology* 21:333-337.

Priestley, H.A. 2003. *Introduction to Complex Analysis*. 2 ed. Oxford, UK: Oxford University Press.

Robert, A. 1982. L'Acquisition de la notion de convergence des suites numériques dans l'Enseignement Supérieur. *Recherches en Didactique des Mathématiques* 3 (3):307-341.

Spivak, M. 1967. *Calculus*. New York: W. A. Benjamin, Inc.

Stephenson, G. 1973. *Mathematical methods for Science students*. 2 ed. Essex, UK: Longman Scientific & Technical.