Children's understandings of algebra 30 years on

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In this paper, we outline the design and method of Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS). Phase 1 consists of a large-scale survey of attainment in algebra and multiplicative reasoning, using test items developed during the 1970s for the Concepts in Secondary Mathematics and Science (CSMS) study (Hart 1981). This will enable a comparison of children's current attainment with that of 30 years ago. Phase 2 consists of a collaborative research study with 8 teachers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment. In June 2008, tests were administered to a sample of 3000 children in each of Years 7, 8 and 9. In addition, attitude questionnaires were administered. A sub-sample of these children will be followed longitudinally and tested in 2009 and 2010. A further cross-sectional sample will be administered in 2009.

Keywords: secondary; pupil learning; assessment.

Introduction

Over the past 30 years, there has been a great deal of work directed at, first, understanding children's difficulties in mathematics and, second, examining ways of tackling these difficulties. Yet, there is no clear evidence that that this work has had a significant effect in terms of improving either attainment or engagement in mathematics. Indeed, children continue to have considerable difficulties with algebra and multiplicative reasoning in particular (Brown, Brown, and Bibby 2008).

The original Concepts in Secondary Mathematics and Science (CSMS) study was conducted 30 years ago. The study made a very significant empirical and theoretical contribution to the documentation of children's understandings and misconceptions in school mathematics (Hart 1981). In the intervening period, there have been various large-scale national initiatives directed at improving mathematics teaching and raising attainment: e.g., the National Curriculum, National Testing at age 7, 11 and 14, the National Numeracy Strategy and the Secondary Strategy. Many of these initiatives have drawn directly on the CSMS study. During this period examination results have shown steady and substantial rises in attainment: e.g., the proportion of students achieving level 5 or above in Key Stage 3 (KS3) tests has risen from 56% in 1996 to 76% in 2006 and the proportion of students achieving grade C or above at GCSE has risen from 45% in 1992 to 54% in 2006. However, independent measures of attainment suggest that that these rises may be due more to "teaching to the test" rather than to increases in genuine mathematical understanding. Replication results from the science strand of the CSMS study (using a test on volume and density) suggests that, students' understanding of some mathematical ideas as well as the related science concepts has declined (Shayer, Ginsberg, and Coe 2007). Studies at the primary level indicate that any increases in attainment due to the introduction of the National Numeracy Strategy have been at best modest (Brown et al. 2003; Tymms 2004). Results from the Leverhulme Numeracy Research Programme suggest that any increase in attainment at Year 6 is followed by a reduction in attainment at Year 7 (Hodgen and Brown 2007). Further, Williams et al. (2007) find that, following this dip at Year 7, there is a plateau in attainment across Key Stage 3.

The research study

Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) is a 4-year research project funded by the Economic and Social Research Council as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics disciplines. The project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England followed by a collaborative research study with the teacher-researchers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment and attitudes. The project is in its early stages and we are currently analysing the initial survey results. Comparison with the CSMS study will enable us to examine what gains, if any, have been made over the intervening period. The Phase 2 findings will extend the results to children's understandings in group and classroom settings.

Phase 1: The large-scale survey of algebra and multiplicative reasoning 11-14

In Phase 1, we are conducting a large-scale survey of attainment in algebra and multiplicative reasoning and attitude to mathematics, involving both cross-sectional and longitudinal elements. This will use test items first developed during the 1970s as part the CSMS study (Hart 1981). Based on a representative sample of schools and students in England, the survey will provide a comprehensive and detailed analysis of current student attainment in Algebra and Multiplicative reasoning. It will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning enabling us to plot where changes have occurred since the original study. It will extend the CSMS study by linking understanding of concepts and student progression to student attitudes, to teaching, and to demographic factors. Analysis is being conducted using a variety of techniques, extending those used in the original CSMS study with Rasch techniques and multi-level modelling.

The survey will consist of both cross-sectional and longitudinal samples identified using the MidYIS database (Tymms and Coe 2003). Three original CSMS tests (Ratio, Algebra, Decimals) will be administered with some additional items relating to fractions (drawn from the CSMS Fractions test) and spreadsheet items. Piloting indicated that only minor updating of language and contexts was required.

The test items range from very basic to sophisticated, allowing broad stages of attainment in each topic to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding.

Phase 2: The collaborative research study investigating formative assessment

In Phase 2, we are conducting a collaborative research study with teachers, which will indicate how they can best use a formative assessment focus within these curriculum areas to improve student confidence and competence, and thus participation, engagement and attainment. Whilst research suggests the efficacy of formative and diagnostic assessment, there is also considerable evidence that teachers have considerable difficulties implementing these ideas (Watson 2006).

Initially teachers will be supported in interpreting and acting upon the survey results of their students; later they will use classroom-based formative assessment based on the frameworks for learning provided by the tests, and assessment for learning approaches. They will also draw on research-informed approaches to the teaching of these curriculum areas. This study will, first, examine how teachers can make use of existing resources and initiatives to respond to students' learning needs, and, second, develop and evaluate an intervention designed to enable a wider group of teachers with much less support to do this. In the final year of the study, the approach will be implemented and evaluated with a further group of teachers and classes.

The Phase 1 findings will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning to inform the teachers and teacher-researchers in Phase 2 both about their own students and about where they lie relative to the general population.

A central question for Phase 2 will be how the generic approach of formative assessment can be adapted to the particular needs of mathematics teaching and learning. This will be done in several ways. First, the diagnostic results for individual students assessed against the learning and progression framework developed by CSMS will guide teachers in planning appropriate work for students and in further formative assessment. The CSMS tests were carefully designed over the 5-year project starting with diagnostic interviews in order to focus on student progression in understanding of key concepts such as variable and rational number. Second, we will identify and link existing teaching resources into the developmental and diagnostic learning structure provided by CSMS building on and extending our existing work in this area which is underpinned by a combination of Piagetian and Vygotskian theories (Brown 1992). There is extensive research evidence relating to the teaching and learning of both algebra and multiplicative reasoning that can inform this intervention (Bednarz, Kieran, and Lee 1996; Sutherland et al. 2000; Swan 2006; Mason and Sutherland 2002; Ainley, Bills, and Wilson 2005), but these research findings and resources have only made a limited impact on teaching practices in classrooms. The solution lies not in designing yet another resource for the teaching of algebra and multiplicative reasoning, but in supporting the judicious use and interpretation of existing resources by teachers (Askew 1996). Third, we will develop our existing work in this area (Hodgen and Wiliam 2006).

The work to date

In June 2008, tests were administered to a sample of approximately 3000 KS3 students. In addition, attitude questionnaires were administered. These data have been coded and are currently being analysed. A sub-sample of these students will be followed longitudinally and tested in 2009 and 2010. A further cross-sectional sample will be administered in 2009. Phase 2 started in Autumn 2008. We have begun to analyse children's understandings of algebra in group and classroom settings and to compare these with the test results.

Early analysis: acceptance of lack of closure

Since the analysis is at an early stage, in this section we focus on just one item, 5(c), from the Algebra test: If $e + f = 8$, what is $e + f + g$? See Figure 1 for the presentation of this item in the Algebra test.

$$\text{If}\quad e+f = 8$$

$$e+f+g = \ldots\ldots$$

Figure 1: Item 5(c) from the Algebra test.

This item was designed to test whether students would readily 'accept the lack of closure' (Collis 1978) of an expression like 8 + *g*. Students tend to see the expression as an instruction to do something and many are reluctant to accept that it can also be seen as an entity (in this case, a number) in its own right. Thus, if one considers, say, the Y9 students tested in 1976, only 41% gave the response 8 + *g*. First indications suggest that the facility for our current Y9 sample is broadly similar to that in 1976. (See Figure 2 for a rough comparison of the unweighted 1976 and 2008 survey results.)

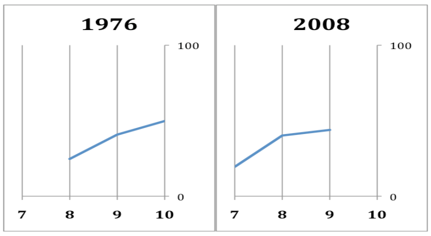

Figure 2: Facilities for item 5(c): $e + f + g$ in 1976 (Y8-10) [n=2820] and 2008 (Y7-9) [n=1810]. Note: in 1976, children were first surveyed at the then equivalent of Y8, because there was relatively little algebra taught to the then equivalent of Y7.

In 1976, statistical software was relatively limited in the size of datasets that could be handled. One interesting feature is the extent to which we can now analyse the full variety of responses as shown in Figure 3. Numeric responses [together with common explanations given by students in interviews] included 12 [4 + 4 + 4], 9 [8 + 1] and 15 [3 + 5 + 7 or alphabet code, *g* = 7].

We recently interviewed several small groups of Y8 students on this item. Their collective meaning making was often quite revealing for us and fruitful for the students. In one group (of four students, taken from a quite high attaining Y8 set) one student had almost immediately offered the response 8*g*, but then, more than a minute later, another student suggested 8 + *g*. Asked which they preferred, one student expressed the 'lack of closure' dilemma very nicely:

> 8*g* sounds more like maths.... than 8+*g*.... which sounds (like a) bit of a sum which you have to work out; but 8*g* just seems like an answer.... in itself.... but 8+*g*, you still think, 'Oh, what will it equal?'.

At the end of the discussion on this item (which ran for nearly four minutes), three of the students opted for 8 + *g*, with the fourth sticking with 8*g*.

Another group (from a high attaining Y8 set) took just a few seconds to decide that the answer had to be 8 + *g*, "because you're not told what *g* is". For them, this property of *g* was unproblematic. On the other hand, it provided an insurmountable barrier for another group. Here, one student felt that $e + f$ (and hence $e + f + g$) "... could be anything ... 2+6, 4+4 ...". Asked what was the best we could do, given that we didn't know the value of *g*, she suggested an answer of 12 (with $e + f + g = 4 + 4 + 4$) as this is "just the easiest one". Another student felt that one needed "to find out what g is and then you could add *g* to 8 to give you the answer". In the meantime, the best one could do was "just guess".

Though this group did not get any further with this item, they exhibited a test-savviness that we also noticed in other groups and which was probably not as prominent in the past. Evidence for this can be seen in this exchange:

Interviewer: So what are we going to do for the answer?

Student 1: Take the simplest one

Student 2: No ... no, 'cos then they'll probably just trick you into thinking that it'll be the simplest one ... They'll trick you into thinking that it's 4+4+4.

Alg q.5c: e+f+g: All Responses - 2008 (Y7, Y8, Y9)

Response	Frequency	Response	Frequency	Response	Frequency	Response	Frequency
	88	2	2	8	30	ANYTHING>	1
?	10	2+2+4	1	8+	2	CANT TELL	2
?UNABLE	1	20	3	8+5	1	DEPENDSON	1
(8+G)	1	200	1	8+8	1	DOESNTTEL	1
0	2	21	1	8+9	1	DONTKNOW	1
1	2	210	1	8+E+F+G	1	E	1
1 12	1	24	1	8+G	634	E+8	1
10	65	29	1	8+G OR 12	2	E+E+G	1
10+E+F	1	2EFG	1	8+G OR G+	1	E+F+G	2
100	1	2G	1	8+G=E+F	1	E+F+G=N	1
11	16	3	6	8+N	1	E+FG	1
12	386	3EFG	1	84G	1	EFG	5
12?	1	4	1	85	1	F+E+G	1
12.5	1	4+4+G	1	8E/F/G	1	G	2
120	1	4+G	1	8EF+G	1	G+5	1
12KG	1	402	1	8F	1	G+8	85
13	14	46	1	8F+G	1	GTE	1
13.5	3	48	1	8FG	1	H	3
14	10	58+G	1	8G	106	N	1
15	62	5EG	1	8G OR 8+G	1	STUPIDQUE	1
16	40	6	3	8G+8	1	TRICKQUES	1
16?	1	60	1	8N	1	WHAT SI G	1
17	3	64	2	8TG	2	WHATS G	2
18	10	7	3	9	137	X	2
1E+1F+1G	1	70	1	9+8	1		

Figure 3: All responses and frequencies for item 5c: $e + f + g =$? [n=1810. Frequencies greater than 1% of sample highlighted.]

Conclusion

As we have already noted, the analysis is at an early stage. We look forward to reporting more extensive findings at future BSRLM day conferences.

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