Hewitt, D. (2006) Proceedings of the British Society for Research into Learning Mathematics 26(2) June 2006

STRUCTURING STUDENTS' AWARENESS OF GENERALITY IN WHOLE CLASS DISCUSSION

Helen Drury

Centre for Mathematics Education, Open University

While leading whole class conversations, as a teacher, I aim to remain aware of students' powers to appreciate and express generality. In this paper I focus on a whole class conversation that took place during a game of 'algebra bingo'. I consider the various types of generalities and examine the extent to which students in the class might be aware of each generality.

INTRODUCTION

Much of my focus as a teacher, whether considering language use, task design, or lesson structure, centres on structuring awareness. I see it as my responsibility to help students focus their attention on those things that I think are particularly worth attending to. I have recently been concerned with the specific effect that using technical mathematical language can have on focussing students' attention on generality. Activities where skills are practised, such as 'algebra bingo', can be developed so that they also offer opportunities to explore generality. The discussion surrounding such practice activities may appear more relevant and interesting as a consequence of being triggered by such a game. In developing students' sense of generality, for example, their attention can be drawn to which numbers in an example are structural, and which are particular, or whether a statement is sometimes, always or never true.

When examples are used to illustrate a whole space of possibilities, it is important that students appreciate this general space, and are able to generalise. Krutetskii (1976) observed that the more successful mathematics students in his study were those who could generalise 'on the spot', on the basis of an analysis of just one phenomenon. Following this argument, it would seem desirable for mathematics classrooms to be places where all students become better able to see the general through the particular (Whitehead, 1932). The discussion of generality is thus a central part of mathematics education, and the role of the teacher in promoting and guiding such discussion is worth serious consideration.

ALGEBRA BINGO

7	6	9
11	15	1
2	12	3

The whole-class 'conversation' that follows took place during a game of 'algebra bingo'. Each student drew a 3-by-3 grid and filled it with their choice of integers from 1-15 inclusive (see diagram for example). At the start, students decided that $a = 13$ and $g = 4$. An algebraic expression was written on the board using these letters, and students calculated the value and crossed off the answer in their grids, if they had it. The winner was the first student to cross out all 9 of their numbers.

My first three questions were:

$2(a - 8)$		10
$3(7 - g)$	with corresponding answers of	9
$2(a - 9)$		8

Gender-preserving pseudonyms have been used for each student. '//' indicates one speaker being interrupted by another.

1	Me:	All the questions are having brackets today. I'll start, let me know if you want to suggest a question.
2	Gina:	You're doing a pattern Miss. We'll stop thinking!
3	Me:	Oh yes. But if I carry on like that I'll have to do 7, which seems tricky.
4	Laura:	You could put 7 in front.
5	Sam:	You'll have to put 7 in front.
6	Me:	Will I?
7	Gina:	Or you could put 1, but//
8	Sam:	//That would be silly, you don't need to multiply it by 1.
9	Natalie:	Multiplying by 1 doesn't do anything.
10	Me:	$7(a\text{-}12)$ Does anyone want to make up a question?
11	Laura:	I want *a*. Can I have *a*?
12	Me:	No. I want brackets today.
13	Laura:	Ok. Put it in brackets! Can you do that?
14	Me:	[Writes (a)] I suppose it would mean the same as *a*, but it doesn't count.
15	Chris:	[To Laura] Put a 1 in front of the brackets.
16	Rebecca:	[To Laura] Do $(a - 0)$ in the bracket.
17	Students:	[Talk amongst themselves. Some of the talk is relevant, some is not.]
18	Me:	Ok. [Pause for quiet]. Let's think about this together. How many different ways can we make 13?
19	Chris:	1, *a* in brackets.
20	Me:	[Writing $1(a)$]. What do you think?
21	Rebecca:	Aren't the brackets supposed to be for doing something? Like $a - 0$?
22	Chris:	Or *a* times 0.
23	Me:	[Writing $1(a - 0)$ and $1(a \times 0)$ on the board]. Are they both the same?
24	Laura:	*a* times 0 is 0.
25	Chris:	Oh yeah.
26	Me:	Are they both the same?
27	Chris:	No, my one doesn't work. $a + 0$ would though.

Several students then put their hands up to suggest other expressions, and the game continued.

ANALYSIS AND INTERPRETATION

Analysis of this transcript led to an increasing appreciation of a variety of categories and levels of generality contained within it. The generalities examined below are those that I can perceive in the discussion, but another reader may discern further examples, or dispute my selection. The same can be said for the students in the classroom, and the extent to which they are aware of the generality being expressed. Whilst a teacher striving to encourage students to think mathematically and to form generalisations may be pleased to note an increase in the 'presence' of such occurrences in their lessons, such generality needs to be present in some sense for the students as well as for the teacher.

Generalisations about… algebra

Students appear to be using general rules about algebra in the extract. Rather than ask for the number she wanted (13, in line 11), for example, Laura said "I want *a*. Can I have *a*?". This seems to be an application of the rule that a letter can represent a number. For her, it seems, for this game, *a* and 13 have become interchangeable. The understanding that a letter represents a number is considered to be an essential foundation of algebraic understanding (Rosnick and Clement, 1980).

There is some ambiguity about whether students are *using* a general rule in a particular case, or moving from the particular to *express* a general rule. Because the students are using algebra, some of their applications of generality may be interpreted as expressions. For example, when Chris and Rebecca suggest ways that Laura can achieve the answer *a*, their suggestions 1(*a)* and (*a* - 0) would equal *a* for *any* value of *a*. They may be saying more than that multiplying 13 by 1 gives 13, or that 13 – 0 is 13. We cannot tell whether they, or the other students, are aware of this. A listener could interpret their suggestions as being true only when *a* = 13.

In line 5, Sam suggests that an answer of 7 can only be achieved if a 7 is put in front of the brackets. With this distinction between 7 as *a possible* answer, and the *only* way to get 7 as an answer, Sam appears to demonstrate some general sense of 7 as a prime number. His expression could perhaps be interpreted as '7 is the only interesting factor of 7', which seems to be a more general statement than '7 is a factor of 7'. Natalie appears to be justifying Sam's particular statement about multiplying a bracketed expression by 1 to obtain the answer 7 with the expression of a general rule:

8	Sam:	That would be silly, you don't need to multiply it by 1.
9	Natalie:	Multiplying by 1 doesn't do anything.

With her use of the present tense here, Natalie seems to imply the general 'multiplying any number by 1 doesn't do anything' rather than the more particular 'multiplying this number by 1 wouldn't do anything'.

Uncertainty about what type of generality is being expressed, which I have experienced while analysing the transcript, might also be experienced by students trying to make sense of the discussion for themselves. They may also be uncertain about whether a general claim is true. With the intention of encouraging students to think about each others' conjectures, and to realise that the truth or falsity of mathematics can be determined without an external authority (the teacher), I tend not to correct students' imperfect conjectures immediately. A student referring to a general rule might, if given space and time, provide a prompt for other students to test the conjecture and develop their understanding. It would be debilitating, however, if deep consideration were required on every occasion a generality was implied or inferred. Although a student may well generalise the rule, and even verbalise the rule, for themselves, the teacher might promote the process by emphasising its importance. Some generalities will be more useful than others, and the teacher may be able to indicate to the student which these might be.

Generalisations about… the game

I write 'game' here, but few of the students appear to see competition as the main purpose of the activity. I find myself frustrated when the activity is interrupted by students telling us "I've got a full house!". Some of the students seem to feel the same. When someone won and we stopped, for example, Laura quietly said "I got a full house ages ago". She apparently didn't think it was interesting enough to tell us about at the time. It is tempting to conclude that the game is an irrelevance, but I suspect that it acts to focuses students' attention in the first place, prompting an initial act of engagement.

It is unclear how students distinguish between the rules of mathematics and the rules of the game. When Laura asks, in line 13, whether she can put *a* in brackets, it is unclear whether she is clarifying the rules of the game or of the mathematical world. Likewise, when Rebecca questioned whether you would or could have brackets that weren't for "doing something" (line 21) she may have been asking about the rules of conventional algebraic notation, or the rules of the game. As I had insisted (line 1) that all expressions must have brackets, Rebecca may have been interpreting my game rule as 'all expressions must have brackets that do something'. Her contribution can be seen either as a reminder of the rules of the game, or a general statement about the meaning and role of brackets in mathematics. In either case, her observation is a non-trivial one, especially given that in written language brackets are often used to designate the inessential.

It is possible to distinguish levels of convention or arbitrariness in mathematics, of which games, tasks and exercises are perhaps the most arbitrary and transient. Students' awareness of the level at which a generality is operating seems crucial to their understanding. The arbitrary nature of general rules in classroom activities often acts to limit the range of permissible change (Watson and Mason, 2005) associated with an aspect of mathematics. "Pick any three numbers", for example, may always mean (for a given group, with a given teacher) "pick any three 2-digit integers". If many activities require this sort of number, then time is saved by establishing 2-digit integers as the range of permissible values for these questions. If students understand that this is a classroom convention, and are still aware of the huge range of numbers that is actually available, then this practice is unproblematic. Unfortunately, this is unlikely to be the case.

I experienced the effect of falsely reducing the dimensions-of-possible-variation for myself during the lesson, when I expressed the difficulty of ‘making 7’ within the restriction ‘with brackets’. I was restricting my interpretation of ‘with brackets’ to only those expressions of the form $a(x + y)$. In retrospect, I could have introduced 2(g – 2) + 3, or an equivalent, thereby allowing for many more possibilities. None of the students suggested an expression of the form $a(x + y) + b$, which suggests that, for this activity at least, the dimensions of possible variation of an expression with brackets did not include such a form.

Line 10 might have been a good opportunity to ask “Could I put anything else, or just 7 and 1?”, and discuss prime numbers and factors. I could have let the students think of alternative ways to make 7, but I felt that the pace of the activity would suffer. Many classroom activities have an extra purpose, alongside that of learning a particular topic. With tests the focus can move from learning mathematics to ‘getting a good mark’, while with games it might move to ‘winning’. While these objectives are generally seen as less important than the mathematics, they may offer students an incentive to access the task.

Generalisations about… behaviour and purpose

Many of the students’ suggestions and assertions give insight into more than their knowledge of mathematics. Just as rules about mathematics can be formed on the basis of several particular examples (or even just one), rules about behaviour are being formed and revised based on particular instances in lessons. The general understandings of a group at a time combine to constitute a community of practice (Lave and Wenger, 1991) or a local community of practice (Winbourne and Watson, 1998). These generalities can be discerned from particular instances.

One such generality concerns the role of errors and misconceptions in maths lessons. Chris’s readiness to accept that “my one doesn’t work” (line 27) and to suggest an alternative is not exceptional in this group. Chris’s comment can be seen as a particular example of it being ‘safe’ to admit being wrong. The comment may be used by other students to create a general rule, leading to a conjecturing atmosphere.

A second generality that I can perceive in the discussion is the rule that ‘when other students are talking amongst themselves, it’s ok for us to do that too’. A large proportion of the students were offering questions, answers, or other contributions. They were otherwise silent. This is perhaps the nearest I can come, as an observer, to claiming that they were ‘listening’. The social contract that ensures that students listen during such discussions seems to break down in line 17. This particular occurrence is an example of a general tendency for students to talk amongst themselves if I appear to have taken a step back from the discussion. As their teacher, I might be able to share with them explicitly the objective that they listen carefully to discussions in which I am not participating.

This kind of meta-communication (communicating about how we communicate) seems to have had an impact in other areas. In line 2, for example, Gina appears to be discouraging me from making it too easy, as they will ‘stop thinking’. In other lessons also, these students appear to see thinking as the central purpose of mathematics lessons. I believe that this is partly due to the emphasis that I placed, especially at the start of the year, on the value of thinking.

REFLECTIONS

The bingo game offers an illustration of how tasks with the principal objective of practising skills can provide an opportunity for the discussion and development of ideas. The extent to which students take up this opportunity is greatly shaped by the role played by the teacher. In just a few minutes of classroom discussion, generalities were expressed concerning algebra (cognitive generalisation), the rules of the game (social generalisation), and behaviour (behavioural and possibly affective generalisations) in mathematics lessons. If there is a value for students in being in the presence of these abstract ideas, then such a benefit might be increased by emphasising the generalities, and drawing students’ attention to what is happening in the discussion. Ellis (2005) found that teacher requests for justification led to “more productive” generalisation, and that repeated use of “why” led to improved, higher-level restatements of the generality. Unless the teacher questions and probes for this justification, the possibility remains that surface-level observations will have the same status amongst listening students as those that are much more profound.

REFERENCES

Ellis, A. (2005) ‘Justification as a support for generalizing: Students’ reasoning with linear relationships’. *Proceedings of the 27th Annual Meeting of PME-NA*, Virginia Tech, October 2005.

Lave, J. and Wenger, E. (1991) ‘Situated Learning: Legitimate Peripheral Participation’, *Cambridge Journal of Education*, 30(2), pp. 275-289.

Krutetskii, V. (1976) *The Psychology of Mathematical Abilities in Schoolchildren*, University of Chicago Press, Chicago.

Rosnick, P., and Clement, J. (1980) ‘Learning Without Understanding: The Effect of Tutoring Strategies on Algebra Misconceptions’, *Journal of Mathematical Behavior* 3, No. 1.

Watson, A. and Mason, J. (2005). *Mathematics as a Constructive Activity: learners generating examples*. Mahwah: Erlbaum.

Whitehead, A. (1932) *The Aims of Education and Other Essays*. Williams and Norgate, London.

Winbourne P. and Watson, A. 1998, Participation in learning mathematics through shared local practices. In A. Olivier and K. Newstead (Eds.) *Proceedings of the 22nd International Group for the Psychology of Mathematics Education*, Stellenbosch, SA, Vol. 4 p177-184.