THE DISCOURSE OF 'SUBTRACTION AS DIFFERENCE'

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For more than 30 years, UK Early Years discourse has referred to 'subtraction as difference' in contrast to 'subtraction as take away' (see, e.g. the Teaching Programme for Year 1 in the National Numeracy Framework). An offshoot of a study of videotapes of trainee primary school teachers' lessons has been clearer thinking (on my part) about the rationale for my long-standing unease with this use of 'difference'. I explain that objection in this paper, and plea for a change in the way UK practitioners refer to this aspect of subtraction.

INTRODUCTION

Carpenter and Moser (1983) identify four broad types of subtraction problem structure, which they call *change, combine, compare, equalise*. These find their way into the UK practitioner literature as different subtraction 'models'(e.g. Haylock and Cockburn, 1997). Two of these problem types are particularly relevant to this paper. First, the *change* problem type, exemplified by Carpenter and Moser by: "Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left" (p. 16). This involves an *action* on and transformation of a *single* set (Connie's marbles). The original set and the transformed set are of the same kind (here, sets of marbles). The UK practitioner language for this is subtraction as 'take away' (DfEE, 1999a, p. 5/28).

Secondly, the *compare* problem type, one version of which is: "Connie has 13 marbles and Jim has 5 marbles. How many more marbles does Connie have than Jim". (Carpenter and Moser, 1983, p. 16). This subtraction problem type has to do with situations in which two sets (Connie's marbles and Jim's) are considered simultaneously - what Carpenter and Moser describe as "static relationships", involving "the comparison of two distinct, disjoint sets"(p. 15). Indeed, the sets need not even be of the same kind e.g. "There are 5 cups and 3 saucers etc.".

The National Numeracy Strategy (NNS) reflects the tradition of UK practitioners in referring to the compare structure as 'subtraction as *difference'*. Specifically, the teaching programme for Reception includes:

> Begin to relate subtraction to 'taking away' and counting how many are left

progressing in Year 1 to:

> Understand the operation of addition, and of subtraction (as 'take away', 'difference', and 'how many more to make') and use the related vocabulary.

The exemplification section of the *Framework* makes it clear that Year 1 teachers are expected to use the language of 'difference' in the classroom.

> Year 1 pupils should … respond rapidly to questions … such as … "What is the difference between 14 and 12?" (DfEE, 1999a, p. 5/28)

In the above quotation ("What is the difference *etc*?") the term *difference* is that name of the outcome of *any* subtraction operation, on a par with *sum*, *product* and *quotient* in relation to the other three arithmetic operations. Historically, in English primary education at least, attention to *difference* has been strangely privileged over the other three outcome-names (Rowland, 1995), although this anomaly is addressed (if not actually redressed) in the guidance on vocabulary for the National Numeracy Strategy (DfEE, 1999b) which gives *sum* and *difference* in the checklist for the Reception year, with *product* and *quotient* in Years 3 and 4 respectively. But *difference* is a trap for the unwary: pupils' problems associated with the ambiguity of the word are well-documented (Pimm, 1987; Durkin and Shire, 1991; Rowland, 1995). This is a case of 'polysemy', in that pupils need to discern the restricted mathematical use of a word whose everyday meaning is more comprehensive.

A LESSON ON 'SUBTRACTION AS DIFFERENCE'

Recently, with colleagues in Cambridge, I have been scrutinising 24 videotaped mathematics lessons taught by primary PGCE students on their final placement. In one of them, Naomi is teaching a Year 1 class. It is clear from her lesson plan that she intends to address 'difference' both conceptually and linguistically. That is to say, she wants the pupils to learn to perceive subtraction in terms of comparison, and to be able to answer appropriately questions about the difference between two numbers. Her plan suggests that she is aware of the two models (problem structures) of subtraction discussed above, and the need for children to learn both. In her introduction, she arranged the frogs into two rows, to facilitate *comparison* of the two sets.

The following extract shows that the ambiguity of 'difference' is manifest from the outset.

Naomi: Right. I had four frogs, so I was really pleased about that, but then my neighbour came over. She's got some frogs as well, but she's only got two. *How many more* frogs have I got? Martin? [emphasis added]

Martin: Two.

Naomi: Two. So what's the difference between my pond and her pond in the number of frogs? Jeffrey.

Jeffrey: Um, um, when he had a frog you only had two frogs.

Naomi: What's the difference in number? This is my pond here, this line, that's what's in my pond, but this is what's in my neighbour's pond, Mr Brown's pond, he's got two. But I've got four, so, Martin said I've got two more than him. But we can say that another way. We can say the difference is two frogs. There's two. You can take these two and count on three, four, and I've got two extra.

First, Naomi poses the comparison problem in terms of "how many more?", and Martin is able to respond correctly to this formulation. Her next question seems to

anticipate the ambiguity problem in that she asks for the difference *in the number of frogs*. Whilst Jeffrey's reply is indeed about numbers of frogs, the word 'difference' has not cued him as intended, and Naomi has to be more explicit ("we can say that another way") about the connection with the earlier "more than" problem. Perhaps her introduction of the word is too abrupt, failing to anticipate the complexity of the particularities of the enculturation in which she and the pupils are engaged. But would it not have been possible to engage them in comparison of the two sets of frogs without bringing in the word 'difference' at all?

There are other instances of children clearly having difficulty with making the desired responses to the difference problems, the following being another example in the Introduction to the Main Activity:

Naomi: Bill's got six frogs, and Martin's got four frogs. How many more does Bill have? Ratna?

[no reply]

Martin's got four frogs, one, two, three, four, but how many more does Bill have? How many more? Leo, can you see? How many more does he have?

Leo: Um, six?

Naomi initially responds by saying:

> OK, let me explain it this way. Right, looking at me, thank you, Jeffrey, thank you. Martin's got four and I want to know how many more Bill's got. Bill's got four as well, but then he's got the two extra ones. So, what is the difference?

Jared offers six. Naomi restates the problem, and Stuart answers "two".

'DIFFERENCE' OR 'COMPARISON'?

First, let me make it clear that 'difference' is being used in two senses:

1. As the general name for the outcome of any subtraction. This is the linguistic/mathematical use; it lies within the mainstream of the discourse of mathematicians (unlike, perhaps, *minuend* and *subtrahend*!). I am in no doubt that pupils 'should' learn to comprehend and use the word mathematically - though not necessarily in Year 1. In any case, there is no special imperative for the name of this particular operation - I'd also want pupils' to learn to use *product* and the proper (additive) use of *sum*, though we seem to have done without them for quite a long time (see e.g. Hardcastle and Orton, 1993).

Incidentally, the use of *difference* to mean *positive* difference is commonplace (Abdelnoor, 1979). In a short discussion of subtraction as comparison, Pamela Liebeck (1990) writes:

> Instead of saying 'five is three and two more', some texts for children say, 'The difference between five and three is two', and record this sentence as '5-3=2'. The danger of this approach is that since the difference between 3 and 5 is also two, children might logically but incorrectly record '3-5=2'. (p. 49)

It might be inferred from this that Liebeck would support my view that there is no urgency to introduce this use of 'difference' in the primary years.

2. As an alternative name for one of three (or perhaps four) possible word problem structures - the *comparison* structure. This is the conceptual/didactical use, and belongs firmly in the realm of teachers' pedagogical knowledge and professional discourse. It would have no particular claim to inclusion in a book on mathematics *per se*. I would propose that it would be better avoided in official curriculum documents (such as the NNS Framework) and other kinds of pedagogical literature, since *comparison* does the job just as well, without the attendant confusion of the two uses itemised here, and the lexical ambiguity (polysemy) inherent in the first. This would be consistent with the practice of many authors outside the UK e.g. Carpenter and Moser (1993).

THE PROVENANCE OF 'DIFFERENCE'

So why did we start using 'difference' as the name of the *comparison* subtraction structure in the UK? It is not easy to be definite how and when this came about, but one useful reference is the teacher's manual for the highly-influential *Mathematics for Schools* (Fletcher, 1971) primary text book series. Indeed, the 1970s and 1980s were the decades of 'Fletcher maths'. In a section titled *Comparison and 'take away'*, Fletcher describes comparison in terms of matching the elements of two sets. Some elements of the larger set remain unmatched. Fletcher writes:

> The cardinal number of this unmatched subset denotes the *difference* between the cardinal number of Set A and Set B. In determining a *difference* we compare a set of objects by matching its members with another set of objects. (p. 9, emphasis in the original)

It is clear that Fletcher is associating the word 'difference' with comparison in order to *distinguish* it from take-away, although the grounds for doing so are not made explicit. The association of the two types of use (conceptual and linguistic) can be seen in some UK primary mathematics teaching handbooks, for example:

> Story 2 introduces […] the *comparison* structure. […] When comparing two sets we may ask 'how many more in A?' or 'how many fewer in B?' or 'what is the difference between A and B?' (Haylock and Cockburn, 1997, p. 39).

However, Williams and Shuard (1976), in their classic primary mathematics handbook avoid use of 'difference' in either sense:

> The language of the practical situations in which children use the idea of subtraction is even more varied than for addition. [...] In each situation there is a dominant idea:
>
> i) taking away, as in 'John has 5 sweets; he eats 2; how many are left?' *or*
>
> ii) inverse or complementary addition, as in 'What must be added to 2 to make 5?' *or*
>
> iii) comparison, as in 'How much more is 5 than 2?' (p. 82)

In the USA, use of 'difference' in the conceptual sense clearly pre-dates Fletcher. In a research paper concerning pupils' performances on subtraction word problems, Schell and Burns (1962) wrote:

> Subtraction has been termed as possessing a "triple nature". By triple nature is meant the "take away" situations; "how many more are needed" situations; and "comparison" or "difference" situations. Sometimes these types of subtraction situations are given the names take-away subtraction, additive subtraction and comparative subtraction (p. 214)

Note, however, that Schell and Burns settle for 'comparison' as their way of referring to the third "situation". In a USA elementary mathematics manual published in the 1960s. Starr III (1969) describes four types of "subtraction experience situations".

> 1. Take-away situations, to find the remainder e.g. 6 birds on a wire, 4 flew away, how many were left?
>
> 2. Comparing two numbers, to find how many more or less in one (difference) e.g. 6 birds on one wire, 4 on another, how many more/less birds are on one wire than another?
>
> 3. How many more are needed, to find the difference between the number we have and the number we need e.g. 4 birds on a wire, how many more are needed for 6
>
> 4. Change-making, the difference between purchase price and the amount given. (Starr III, 1969, p. 128)

The use of 'difference' in the characterisation of situations 2., 3. and 4. is arguably linguistic in kind. It is not possible to tell whether the conceptual use is intended in the parentheses in 2.

A very recent USA elementary mathematics teaching manual (Jensen, 2003) brings a unusual degree of formality to the *genre*, this time with no mention of 'difference'.

> Definition 2.15 (Comparative subtraction).
>
> If a and b are whole numbers, then a-b is that whole number c for which $b+c=a$.
>
> Definition 2.16 (Take-away subtraction).
>
> If a and b are whole numbers, then a-b is that whole number c for which $c+b=a$.
>
> Theorem 2.18
>
> If a and b are whole numbers, then a-b in comparative subtraction is equal to a-b in take-away subtraction.[1] (pp. 47-48)

[1] Regarding each of the two definitions, the pedant might ask how we know that c exists and, if it does, that it is unique. For a British mathematics educator, this book is a cultural experience. Readers may wish to know that Jensen's proof of Theorem 2.18 is as follows, in essence: addition is commutative, so if $b+c=a$ then also $c+b=a$.

SUMMARY AND A PROPOSAL

In this paper, I have drawn attention to what I regard as a problematic feature of language use in primary mathematics. Namely, that 'difference' is being used *both* as

the general name for the outcome of any subtraction (linguistic/mathematical use)

and

an alternative name for one of several possible word problem structures - the *comparison* structure (conceptual/didactical use)

The use of 'difference' in the second sense is admittedly very natural. In comparing two sets, one is asking about ways in which they are different. Such differences may be non-mathematical (e.g. kind, colour) or mathematical but non-cardinal (e.g. even/odd, prime/composite). Such differences certainly merit discussion and exploration. The second use above restricts 'difference' to the quantification of comparison of numerosity in two distinct, disjoint sets, unnecessarily invoking the first use.

My **proposal**, then, would be as follows:

- use 'comparison' (not 'difference') in professional discourse (e.g. NNS documentation, teachers' handbooks *etc*) for the name of the comparison structure;
- focus on 'how many more/less?' questions (as opposed to 'what is the difference?') when comparing the cardinality of numbers and sets in KS1.

I recognise that these proposals are both radical and controversial, and that "living with complexity" is often a good thing. I would welcome reactions and comments.

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