LIMIT - A PROOF-GENERATED CONCEPT

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This paper examines the components of the modern definition of the limit of a sequence, in terms of their historical genesis. In classical Greek times it was recognised that limiting arguments must be pursued with inequalities. With Newton the use of a quantity 'as small as one may wish' (our modern ε) was combined with inequalities. It was Cauchy who recognised that smallness was to be achieved 'with sufficiently large N'. In each case these components of the modern definition emerge in proofs not in definitions. None of these components conventionally play a part in pre-university proofs and this may explain why the formal definition of limit is regarded as an 'epistemological obstacle'.

The phrase 'proof-generated concept' originates with Lakatos [1976, page 89]. Unlike Euler's theorem on polyhedra, the limit notion has not developed by the invention of counter-examples, but by a widening of the problems addressed and the consequent increased generality required in methods of proof. In classical Greek times, limits, as such, were not investigated. But there were lengths, areas and volumes to which conventional methods of measurement did not apply. The use of inequalities by Euclid and Archimedes generated methods for measuring some of these awkward quantities. In the early 17th century Fermat and others became highly skilled in applying what they called 'the method of Archimedes'. The arguments in Newton's *Principia* (1687) are largely geometrical arguments about limits, and it is with Newton that the word 'limit' appears in something near our modern sense. In his proofs Newton uses the notion of a quantity 'as small as one may wish'. Cauchy (1821) gave only a verbal definition of limit, but used something indistinguishable from our modern definition in his proofs.

To get a sense of what an historical view of concept-development may be like, and before we launch into our detailed study of limits, it may be helpful to listen to a quotation from Judith Grabiner (an excellent historian of 18th century mathematics) on derivatives:

> The derivative was first *used*; it was then *discovered*; it was then *explored and developed*; and it was finally *defined*. [Grabiner, 1983]

used: Fermat and others before 1650
discovered: Newton and Leibniz 1666 - 1685
explored and developed: 18th century

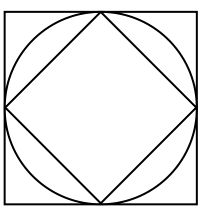
defined: Lagrange(1797), Cauchy (1823) and Weierstrass (1860s)
The point Grabiner is making is that the formal definition (of derivative) was reached after a lengthy mathematical process, and I suggest that, compared with that of derivative, the definition of limit was reached after a much longer process.

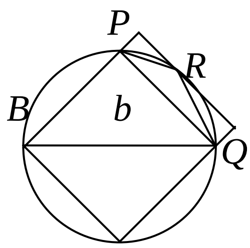
Let us look at the standard 20th century definition of the convergence of a sequence.
The sequence (a_n) tends to a means
✷ ✹ ✷✶ ✶ ✷ ✹
Given $\varepsilon > 0$, there exists an N, such that $a - \varepsilon < a_n < a + \varepsilon$ when $n > N$.

The components here, emerge in proofs constructed by
- ✶ Euclid (*c*.300 BC) and Archimedes (*c*.250 BC) - inequalities
- ✷ Newton (1687) - ε
- ✹ Cauchy (1821) - *N*

The use of inequalities by Euclid and Archimedes
The basis of all Greek 'limit-like' arguments is Euclid X.1 which is equivalent to the Archimedean Axiom. In particular it is the basis for the proofs in Euclid Book XII relating to the area of a circle, the volume of a pyramid and the volume of a cone.
Euclid X.1 *Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.*

We see how Euclid X.1 is applied in Euclid XII.2.
Euclid XII.2 *Circles are to one another as the squares on their diameters.*





To Prove: $A/B = a^2/b^2$. Proof by contradiction. If $A/B \neq a^2/b^2$, then either $A/B > a^2/b^2$ or $A/B < a^2/b^2$.   Suppose $A/B < a^2/b^2$, then $A/S = a^2/b^2$ for some $S < B$.
Now, to apply Euclid X.1, we take the greater magnitude to be the area B of the circle and the lesser magnitude to be $B - S$.

Area of square on diameter $b > \frac{1}{2} B$, since this square is half the area of a square circumscribed about B.
B – square on diameter b = 4 circular segments on side of square. If a side of this square is named PQ and R is the mid-point of the circular arc PQ, then the triangle PQR is more than half the circular segment PQR. Likewise for all four sides. Now consider the regular octagon with diameter b. (regular octagon – square) > ½(B – square). By same argument, (reg.16-gon on diam. b – reg. octagon) > ½(B – octagon), and (reg. 32-gon on diam. b – 16-gon) > ½ (B – 16-gon). By Euclid X.1, for some regular polygon $p(B)$ on b as diameter, $B - p(B) < B - S$.
So $p(B) > S$. Now if $p(A)$ is a polygon similar to $p(B)$ inscribed in the circle A, $p(A)/p(B) = a^2/b^2 = A/S \Rightarrow A/p(A) = S/p(B)$, but $A > p(A)$ and $S < p(B)$. Contradiction. Similarly for the other inequality.

The greatest skill in constructing Greek limit arguments was shown by Archimedes who used them to connect the area and circumference of a circle, to find the surface area and volume of a sphere and to solve a host of other problems. [See Dijksterhuis, 1987]
As an illustration we cite his quadrature of the parabola [detail in Fauvel and Gray, page 153] To prove that the area of the parabolic segment, S, bounded by a chord PQ = (4/3) area A of the maximum triangle in the segment, Archimedes, like Euclid, proceeded by contradiction. if $S \neq (4/3)A$, *either* $S > (4/3)A$ *or* $S < (4/3)A$. Each of these possibilities must be contradicted.
Now the maximum triangle in the segment with base PQ has more than half the area of the segment in which it lies, and the same is true of the remaining segments. Archimedes disproves both the inequalities by appealing to Euclid X.1 and making judicious use of the equation
$A + \frac{1}{4}A + (\frac{1}{4})^2 A + \ldots + (\frac{1}{4})^n A + (1/3)(\frac{1}{4})^n A = (4/3)A.$

When we come to Fermat's use of Archimedes' method (1658), Fermat does *not* contradict two inequalities, but says that his proofs could be set up that way but would be unhelpfully long.

I owe what I am going to say about Newton, to a recent paper in *Historia Mathematica* [Pourciau, 2001] comparing what Newton and Cauchy wrote about limits. Traditionally Newton is considered to be vague and Cauchy precise. Pourciau showed that this traditional view depended on which bits of Newton and which bits of Cauchy one read. If you choose quotations appropriately Newton can be made to look quite as precise as Cauchy. Now it is true that the use of the Greek letter ε was Cauchy's choice, but by looking carefully at Newton's proofs one can see him using a quantity “as small as one might wish” with modern precision. There is some discussion of convergent sequences in John Wallis’ *Arithmetica Infinitorum* (1655) which we know Newton studied. But we should remember when thinking about Newton that one of Kepler’s laws was that the focal radii joining the planets to the sun sweep out equal areas in equal times, so in Newton’s diagrams of orbits, time appears in the form of area. In the *Principia* (1687) many of Newton’s arguments concern geometrical limits, like secants tending to a tangent. So let us turn to the *Principia* and look at Newton’s first two lemmas. We will find him using arbitrarily small quantities to establish limits.

<u>Newton</u> *Principia*, 1687 (trans. Motte - Cajori, page 39)

> Those ultimate ratios...are not actually ratios of ultimate quantities, but limits...which they can approach so closely that their difference is less than any given quantity. [Fauvel and Gray, page 394]

Principia, lemma 1 (trans. Motte - Cajori, pages 29 - 30)

> *Quantities, and the ratio of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.*
>
> If you deny it, suppose them to be ultimately unequal, and let *D* be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference *D*; which is contrary to the supposition. [Fauvel and Gray, page 391]

Lemma 2 in Newton’s *Principia* is familiar to modern readers as the proof that a monotonic function is Riemann-integrable. If a given curvilinear area has both inscribed and circumscribed rectangles of equal width, the difference between the areas of the two sets of rectangles is exactly the area of the largest circumscribed rectangle. If the number of rectangles is increased, the width and so the area of the largest is diminished and becomes less than any given area. So the ratio of the two sets of rectangles (and necessarily the curvilinear area also) tends to equality. [See Fauvel and Gray, page 391]

It is Pourciau’s thesis that we owe ε to Newton rather than to Cauchy.

Now let us turn to Cauchy who is generally credited with the modern definition of limit. Here is how Cauchy defined limit, and he used exactly the same wording in 1821, 1823 and 1829, so we can be sure that so far as he was concerned there was no mistake here.

<u>Cauchy’s definition of limit</u>[Cauchy, 1821, page 4]

> Lorsque les valeurs successivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est appelée la limite de toutes les autres.
> [When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to eventually differ from it by as little as one might wish, this latter value is called the limit of all the others.]

The definition has quite an 18th century or even Newtonian ring about it, and, however translated, these are not the words used in the definition given in modern texts. So where does Cauchy earn his reputation as the definer of "limit"? His reputation comes not from his definition, but from his theorems and proofs.

<u>Cauchy's first four theorems with ε, *N* proofs.</u>

[Cauchy 1821, pages 48 - 60]

1. If $f(x + 1) - f(x)$ tends to k, as x increases, then $f(x)/x$ tends to k.
2. If $f(x + 1)/f(x)$ tends to k, as x increases, then $[f(x)]^{1/x}$ tends to k.
3. If $A_{n+1} - A_n$ tends to A, as n increases, then A_n/n tends to A as n increases.
4. If A_{n+1}/A_n tends to A as n increases, then $(A_n)^{1/n}$ tends to A as n increases.

All four were new theorems. Cauchy gives full proofs of the first two and presumes that the third and fourth will follow. Unfortunately the first two are flawed and fail when $f(x) = 1/(1 - x + [x])$, for example. This is presumably why they are not exhibited when the history of limits is being discussed. However Cauchy only claims theorems 1 and 2 in contexts where they *are* valid. To show the first use of what looks like a modern definition, I will cite the first paragraph of Cauchy's proof of theorem 1.

> "First suppose that the quantity k has a finite value, and denote by ε a number as small as one might wish. Because increasing values of x make the difference $f(x + 1) - f(x)$ converge towards the limit k, one can give a number h a sufficiently large value such that, when $x \geq h$, the difference is contained within the limits $k - \varepsilon$ and $k + \varepsilon$."

Here we have all the components of a modern definition, which did not appear as such, in print, until after Weierstrass had begun lecturing (1860). The path from the Greeks through Newton to Cauchy has shown a way to consider 'limit' as a 'proof-generated concept': inequalities with the Greeks, ε with Newton and *N* with Cauchy.

Why is this definition such a problem for our students? One fact seems clear; our courses and texts do not introduce the subject as a 'proof-generated concept'. I have tried to make a list of the places in British school mathematics where limits are implicit or explicit, and this is what I have come up with.

Pre-university results. Area of circle = (radius) × (half circumference); area of circle ∝ (radius)^2; volume of pyramid and cone; $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=1$; $\frac{1}{3}$= 0.33333...; $1 + x + x^2 + x^3 + ... = 1/(1 - x)$; asymptotes to $y = 1/x$:

as $x \to 0^+$, $1/x \to \infty$; as $x \to \infty$, $1/x \to 0$; as $x \to 2$, $\dfrac{x^2-4}{x-2} \to 4$.

I have to admit that in none of these parts of school mathematics which implicitly involve limits do inequalities regularly find a place. Implications with inequalities are incidental to school mathematics. Even such arguments as $n > 5 \Rightarrow n > 4$ or $0 < x < 1 \Rightarrow x^2 < x$ are discomforting at school. Yet the limit definition is *an implication between inequalities* and *that* seems to be an obstacle.

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Lakatos, I.: 1976, Proofs and Refutations, Cambridge University Press

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[The most thorough study of limits at the pre-university level that I know is Hauchart, C. and Rouche, N., 1987: Apprivoiser l’infini, CIACO, Louvain, which is an analysis of pupils’ responses to 14 problems which were published, in 1994, as Some Encounters with Infinity by Manchester Mathematics Resource Group, Didsbury School of Education, 799 Wilmslow Road, Manchester M20 8RR.]