LIMITS, CONTINUITY AND DISCONTINUITY OF FUNCTIONS FROM TWO POINTS OF VIEW: THAT OF THE TEACHER AND THAT OF THE STUDENT

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Abstract: *In this study we show that a primitive idea of limit is inducing an obstacle in the construction of the concept of limit, continuity and discontinuity of functions and an unsuitable internal conceptual structure of the idea of infinity (ambiguity between potential and actual infinity). We analyse a particular teacher's lesson on the concept of limit of functions and relations to continuity and discontinuity of functions, and we explore the ideas one student has on the same concepts through interviews. We show that the teacher's use of natural language, when introducing the idea of limit, emphasise a primitive idea; also, he tries to induce basic algorithm strategies to influence students' learning in an inappropriate way. The wrong strategy followed by the teacher will influence the students' construction of the concept of limit, where the role of infinity is ambiguous, producing a cognitive obstacle as is pointed out in this case.*

Focusing on the idea of cognitive obstacles, we would like to clarify some problems students and teachers have when learning mathematical concepts; we designed a general project, which has been running since 1992, related to the construction of the concepts of function, limit and continuity. The project involves high school teachers of mathematics and students from high school and first year at the University.

At the first stage we studied epistemological obstacles (inherent difficulties that occurred in the development of a mathematical concept) teachers and students meet when dealing with the function concept (Hitt, 1994; Hitt, 1998; Lara, 1995 and Hitt & Planchart, 1998). One of the main results of those papers was that some teachers and students have a strong tendency to think in continuous functions expressed by one formula. We showed in Hitt (1994) that this intuitive idea guided Euler when he wrote his book (1748) *'Introduction in analysis infinitorum'.*

Once we detected several epistemological obstacles related to the concept of function like that mentioned before, we decided to find out if that intuitive idea represent among others an obstacle in the learning of other concepts like that of limit, continuity and discontinuity. **Methodology:** Our study at this stage is about the analysis of written lessons about functions, limit and continuity of functions, provided by 9 mathematics teachers (high school), questionnaires and interviews of students who had just finished high school and were beginning their Engineering studies at the University. The teachers involved in this study agreed with the experiment but only if we could have a blind written lesson.

The instructions to the teachers were to develop a lesson related to the teaching of functions, limits or continuity. They were not allowed to use books or notes. We wanted to know teachers' spontaneous ideas on those concepts.

For the aim of this paper, we would like to discuss a lesson written by one of the teachers in the experiment and the interviews taken from one student in seven sessions during a total time of six and a half hours.

The teacher's lesson we would like to discuss is on the concept of limits of functions and the relations to continuity and discontinuity of functions. The teacher wrote 14 pages and began talking about intuitive ideas on the notion of limit. The teacher wrote: *"To understand limits at infinity and infinity limits, it has to be done from an intuitive point of view, using a numerical approach and graphical interpretation; it is important also to dominate algebraic knowledge .... "* He continues in this way giving examples in real life, e.g. *limit of speed, limit of elasticity of a material or limit on a beach where you can swim safely.* He said that a deduction from the examples is: *"that the limit cannot surpass a mark", "the limit value is not reached", "you can be as near as you wish but not reach it".*

This teacher thought that approaching the concept in this way would help the student to construct a suitable internal conceptual structure related to the concept of limit. But, it seems the teacher is influencing the students in the construction of an inappropriate schema that will be an obstacle to assimilate the concept of limit. The idea the teacher is suggesting is that of limit approached only by one direction (in the domain of a function) and not reaching *"a mark".* It seems that the intuitive idea of potential infinity is related to the independent variable. With this approach, the students will construct a primitive idea of the concept of limit that it is going to conflict with a more advanced conceptualisation of that concept. The primitive ideas that students are going to construct could be similar to those presented by Tall & Vinner (1981) with English students or by Cornu (1983) with French students.

Continuing with the teacher's lesson, he states that: *" ... to understand the notion of the limit of a function it is necessary to work with rational functions to see different cases like the limit of a function could give as a result. a) An integer number, b) a rational number,* c) *Ole, with* c *an integer, then your result is zero, d) c/O,* c *an integer, in that case we have an indetermination, .oo. Also, the limit of a function can give an indetermination like 0/0=00 .... For example*

$f(x)=\dfrac{x^2-9}{x+3}$ *when "x" approaches -3. To find the limit by a numerical approach we need to tabulate the following data*

x	-2.5	-2.9	-2.99	-2.999	-3.001	-3.01	-3.1	-3.5
f(x)	-5.5	-5.9	-5.99	-5.999	-6.001	-6.01	-6.1	-6.5

S
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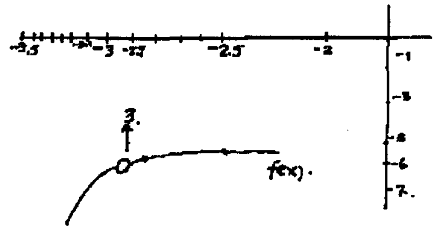
We can observe that the function approximates -6 *by the left and right ... then we can conclude that the limit of the function when "x" approaches* 3 *is* -6. *The graph is*



... In conclusion, the function can have a limit, if it exists, but to that limit the function becomes discontinuous when the limit touches that value".

The graph does not correspond to the function f(x) = x - 3, with x::/:. -3. In this case, does it matter the graph of the function? The teacher seems to think that the idea is what is important, he is trying to express the intuitive idea of a curve with a hole. On the one hand, as we said before, you can see that the teacher is thinking about potential infinity when he is talking about the variable: *"the limit value is not reached", "you can be as near as you wish but not reach it".* Indeed, when using a numerical approach, could the teacher be out of that powerful intuitive idea of the potential infinity? On the other hand, the concept of actual infinity is behind his ideas when he says: *"but to that limit the function becomes discontinuous when the limit touches that value".* That means that the process has already finished, that is that the notion of the actual infinity has been used. Also, the teacher is mixing two concepts, limit and discontinuity. He is not giving a complete idea about discontinuity. He continues:

"The algebraic techniques to evaluate limits are as follows: a) When the result is an integer $\lim_{x\to 1} 2x^3+1$ ***you must substitute directly the value of the variable, then*** $\lim_{x\to 1} 2(1)^3+1=3$, ***b) when the result is a rational number*** $\lim_{x\to 0}\frac{3x+2}{x-6}$ ***the same process like in (a)*** $\lim_{x\to 0}\frac{3(0)+2}{0-6}=-\frac{1}{3}$. ***Also you can have a result with (c), (d) and (e) using an algebraic method to remove the indetermination like*** $\frac{0}{0}$, $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1}$ ***If you substitute directly the value of the variable*** $\lim_{x\to 1}\frac{(1)^2+2(1)-3}{(1)^2-1}=\frac{0}{0}$. ***To remove the intedermination before you do the limit process you do a factorisation*** $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1}=\lim_{x\to 1}\frac{(x+3)(x-1)}{(x+1)(x-1)}$, $\lim_{x\to 1}\frac{x+3}{x+1}$, ***substituting the value of the variable,*** $\lim_{x\to 1}\frac{(1)+3}{(1)+1}=\frac{4}{2}=2$. ***This introduction is useful in studying the notion of continuity of a function."***

We found that this teacher tended to fixate some strategies focusing on algebraic methods to calculate limits, favouring informal approaches to communicate a mathematical idea. His errors show that the teacher has got an unsuitable conceptual structure when calculating limits. The teacher's process of limit is reduced to a substitution. For the teacher, this is something to be calculated $\lim_{x\to 0} \frac{3x+2}{x-6}$ and this is the result: $\lim_{x\to 0} \frac{3(0)+2}{0-6} = -\frac{1}{3}$. At this stage, he is not thinking in a process or properties about limits, but in a substitution. The teacher is showing a behaviour like: *You can always calculate a lim~t, if the result is an indetermination there must be a way to fix it.* We can see that not only has the teacher transformed the concept of limit into a communicable form to communicate with the students, but also that he has got an unsuitable schema that might provoke in the students the construction of a unsuitable schema too. The student errors might be a reproduction of the teacher errors. He has constructed a naIve conceptualisation (in Davis and Vinner sense, 1986, p.281) related to an infinite process where the potential infinite plays a principal role, but the teacher seems not to be aware of it, and that will impede the acquisition of more abstract conceptualisation where the actual infinity could be involved.

In relation to the interviews of the students (10 students), we choose a class of mathematics in a School of Engineering. The first stage of the study with students consisted of the answering of questionnaires related to the concept of function; the results are similar to those presented in the references. Then, in the second stage, interviews with the students were planned.

Let us see one of the interviews with one student. The first question posed to the students was: *"It is true that: ~ = 0.5=0.49 ... " Explain your answer.* The comments given by the students were similar to those pointed out by Tall & Vinner (1981, p. 158-159). For example, a typical answer was: *"0.4999 ... It is not equal to 0.5 but is a value near to 0.5".* We would like to stress here one of the main points related to our study: *The cognitive obstacle showed by the students is because they are dealing with the expression 0.4999 ... from a cognitive structure (scheme) where the idea of potential infinity is playing a major role.* But, *if the affirmation were that 0.4999 ... = 0.5 probably it would meant that the infinite succession has been actually realised and his/her cognitive structure allows the abstract idea of actual infinity.*

In the second question: *The limit deals with the behaviour of a function in: a) a point, b) near a point. Explain.* The same student said:

S. *Well, here in this question if the limit of a function deals with ... My answer is one point.*

S. ***This is because if we have a function $f(x) = x^2 - 1$ when x approaches to…, I do not know, let x be 2, we have the limit $\lim_{x\to 2} f(x) = \lim_{x\to 2} (2)^2 - 1 = 3$.***

In his explanation he constructed a table showing that taking points near to 2 on the left and right of that point the images of the function are approaching 3.

I. ***Can this procedure be used in every case like for example $f(x) = \frac{x^2-1}{x-1}$? What happens when x approaches 1?***

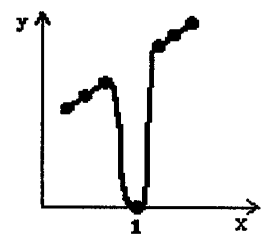
S. *You get zero, it is* (pause) *it is a strong indetermination. Then to do this we need to do a factorisation process.*
I. *Is it zero? Why are you saying it is equal to zero?*
S. *Because* $f(x) = \frac{(1)^2 - 1}{1-1} = \frac{1-1}{1-1} = 0$.
I. *What do you have in the upper part of the fraction?*
S. *Here* (pointing at the upper part of the fraction) *I would obtain zero.*
I. *What is* $\frac{0}{0}$*?*
S. *It should* (pause) *it is zero, isn't it?*

The interviewer asked the student to construct a table and to graph the function when x is near 1. In the table he put for x equal 1, f(x) equal zero.

Table

x	*.9*	*.99*	*.999*	*0*	*1.01*	*1.001*	*1.0001*
f(x)	*1.9*	*1.99*	*1.999*	*0*	*2.01*	*2.001*	*2.0001*

S. *Then here* (pause) *we realise that here* (pointing at x = 1) *the function is zero, we see that the curve is continuous but suddenly is going down to zero and this is like this. And then it continues the graph again and we realise it is not continuous*



We agree with Skemp (1971) when, dealing with understanding, he states: *To understand something means to assimilate it into an appropriate schema. We may achieve a subjective feeling of understanding by assimilation to an inappropriate schema* (p. 43). And when talking about implication for the learning of mathematics, he said: *The central importance of the schema as a tool of learning means that inappropriate early schemas will make the assimilation of later ideas much more difficult, perhaps impossible.* (p. 48)

Discussion:

It seems that this teacher has constructed an idea of infinity in an incoherent scheme. This ambiguity involved in the treatment of the infinity seems to be transmitted to the students. In our student's case, the student's error may be produced by the way this concept is thought. This approach of teaching is hiding the concept of infinity, leaving the students the construction of the concept by themselves.

The teacher and the student have developed the idea of potential infinity related to limits, but the abstract idea of actual infinity seems to be weak in their cognitive structure (scheme). The calculation of limits reduced to a substitution is like a simple algebraic substitution and not like a process that has actually finished. Both, the teacher and the student, are showing a cognitive obstacle. In the student's case we would refer to this phenomenon as a didactic or pedagogical obstacle because it could be induced by the way the concept of limit was thought. The problem is bigger than that, the history of mathematics has shown that in the evolution of the concept inherent difficulties have occurred and because of this the obstacle became an epistemological one. The Greek mathematicians couldn't go through this obstacle. Even though Zeno's ideas were pointing out to this problem. The potential infinity seems to be the only one adopted before G. Cantor.

The student did not realise the function was not defined in $x = 1$. He obtained zero when calculating the limit and he wanted to construct something coherent with his result. The graph of the student is showing that the schema he has got is related to **"a continuous curve and because of the sudden dip to 0 at $x = 1$ of the function, a notion of irregularity of the function is inducing him to think in a discontinuous function."** Because he states *"suddenly is going down to zero ... it continues the graph again and we realise it is not continuous".* This schema is not helping him to assimilate the concept of limit of a function. His idea of discontinuous function is similar to that of Euler (1748), where a "discontinuous curve" was determined by several expressions giving that "behaviour of irregularity". Also, the student's idea of limit process is reduced to a substitution. It is a reproduction of a similar idea of the teacher's conceptualisation of limit.

A new approach of teaching the idea of infinity is necessary to develop in the students a suitable schema where the concept of potential and actual infinity could play a better role in the construction of other concepts like those of limit, continuity and discontinuity of functions.

Acknowledgements: This work was supported, in part, by grant number 26408P-S from Consejo Nacional de Ciencia y Tecnologfa (CONACyT), Mexico.

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