Geometrical Reasoning: a report on the working group at the ICMI Study Conference on "Perspectives on the Teaching of Geometry"

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The ICMI study coriference on "Perspectives on the Teaching of Geometry for the 21st Century" took place recently in Italy. This paper reports on the discussion of one of the conference working groups which considered geometrical reasoning. Four main themes are covered: visual reasoning, geometrical reasoning in context, the meaning of proving in learning geometry, and assessing the range of reasoning ability in geometry. There was general agreement at the conference that more research is necessary in order to effectively address the wide range of issues that were discussed

This latest in the series ofICMI study conferences took place in Italy in September 1995. The theme for the study conference was "Perspectives on the Teaching of Geometry for the 21st Century". It is the first such conference in the ICMI series to consider a specific area of mathematics [1]. The conference was attended by over 70 people from more than 30 countries. For the most part, the conference was organised in working group sessions. There were six working groups:

Curricula change: convened by Lundsgaard Vagn Hansen (Denmark)
Computer technology and software: convened by Iman Os18 (Lebanon)
Social interactions and teaching methods: convened by Regine Douady (France)
Teacher qualifications: convened by Mogens Niss (Denmark)
Geometry and reality: convened by Joe Malkevitch (USA)
Geometrical reasoning: convened by Rina Hershkowitz (Israel)

In addition, there were round table sessions on each of the following three themes:

The evolution of the teaching of geometry
Classical versus non-traditional geometry
Geometry from school to University

The following report is based on the discussions of the working group on *geometrical reasoning*, convened by Rina Hershkowitz (Israel). This working group was by far the largest at the conference with about 30 participants. It met for four sessions each lasting about four hours (some of the other working groups met for half that time). Each of the sessions was organised around a particular theme. The four themes were:

Visual reasoning: led by Rina Hershkowitz (Israel)
Geometrical reasoning in context: led by Mariolina Bartolini Bussi (Italy) The
meaning of proving in learning geometry: led by Celia Hoyles (UK)

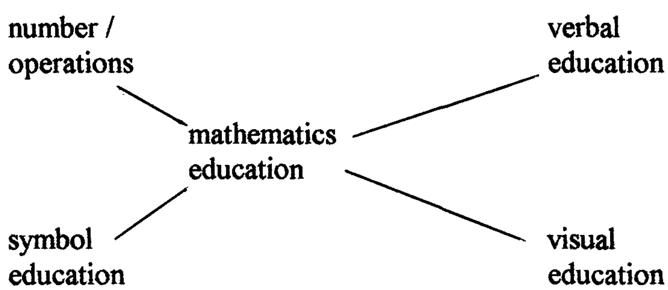
[1] Previous ICM! studies have been published on School Mathematics in the 1990s, The Influence of Computers and Informatics on Mathematics and its Teaching, Mathematics as a Service Subject, Mathematics and Cognition, the Populisation of Mathematics, Assessment in Mathematics Education. To appear shortly are volumes on Gender and Mathematics Education, and What is Research in Mathematics Education? The next ICM! study will be on the role of the history of mathematics. The study conference is likely to take place in France in 1997.

Assessing the range of reasoning ability in geometry: led by Angel Gutierrez (Spain)

During each of the working sessions, a small number of participants were invited to present some ideas on the topic concerned in order to stimulate some wider discussion. The following sections attempt to summarise some of the ensuing discussion. It is both too difficult and too cumbersome to try to attribute the views expressed to particular individuals and likewise too difficult to provide supporting evidence of an acceptable nature. Thus the summary below can only be taken as one *interpretation* of the working group discussion with many of the statements having to be taken at the level of assertion. Evidence to support such assertions was available during the working group sessions and the proceedings of the conference (Mammana 1995) do provide the full text of the conference papers, some of which were summarised during this particular working group. The book that will be produced as a result of this IeMI study is due in 1997.

Visual reasoning

Although it can be argued that we may be experiencing a visual renaissance in mathematics, there is a good deal of evidence that, in mathematics education, visual representation and visualisation are neglected areas. Mathematics education can be considered to involve (at least) the following:



This leads to a number of questions:

what should a visual education in mathematics look like?

What do we mean by visual reasoning? Is it different from geometrical reasoning?

What is the place of systematic language?

These questions lead to various arguments. For instance, there is an argument that visualisation is different from visual reasoning. There is an argument that different terms have different meanings for different people; for instance, it can be argued that, for mathematicians, visual thinking equates with geometrical thinking; or it could be argued that visual thinking is a general term and geometrical thinking is the mathematical part of visual thinking; or that visualisation and mathematisation equals geometry.

Other questions that can be considered include whether visual reasoning is an initial easy stage (as implied, for instance, by the van Hiele levels) or whether it is, in fact, quite difficult. Some time ago, Alan Bishop referred to the ability of "interpreting figural information" (for example, Bishop 1983 p 184). This suggests that orne attention might usefully be paid to identifying appropriate experiences in order to develop this ability. It may be that there is a progressive come-and-go (and perhaps, coordination) between visualisation and verbalisation.

Geometrical reasoning in context

Here the argument is for a contextualised approach to the teaching and learning of geometry in which student activity is driven by an exploration of a *field of experience.* Fields of experience can involve everyday experience (for example, shadows) or mathematical culture (for instance, conics). The function of this approach is to contextualise what can be isolated parts of mathematics and, additionally, to use the history of mathematics in a productive way. Some of the fields of experience that may be worth considering include, in the primary sector, such everyday occurrences as walking, seeing patterns, and measuring how far and how long, and in the secondary sector such contexts as shadows from, for instance, long poles (for example: two vertical poles produce parallel shadows on horizontal ground, what about a vertical pole and a non-vertical pole?). Another context would be linkages similar to the pentograph that students can manipulate.

In all these cases, it requires a social culture in the classroom that values classroom talk and discussion, that values doubt and conjecture and that recognises the mediating effect of tools and tasks.

The meaning of proving in learning geometry

There is currently an ongoing debate about the nature of proof For mathematicians the debate is about the nature of proof in mathematics, amongst mathematics educators it is about the nature of proof in learning mathematics. In terms of the meaning of proving in learning geometry, there is evidence from a number of fields. For instance, there is some evidence from psychology that making some forms oflogical connectives is difficult. This may have implications for the learning of proving. On the other hand, it can be argued that proving is not the same as using logical connectives in terms of the cognitive processes involved. The validity of a conclusion, or of a proof, has a cultural aspect and this involves mediation.

The evidence from research in mathematics education is that students can feel that empirical evidence is sufficient. On the other hand, they can feel that deductive reasoning is not necessarily true for all cases. Mathematicians, it is said, experience the same sorts of feelings.

The function of proof can be modelled in various ways (after de Villiers):
verification (concerned with the truth of statements)
explanation (why it is true)
discovery (of new results)
systemisation (local deductive chains, connecting theorems)
intellectual challenge
communication

The nature of proof in the classroom is effected by the specification of the curriculum. For instance, in the UK. there has been a shift away from 'proving' as *aformal activity* (as exemplified by the US "two-column proof' approach) towards what can be called *data-driven investigations.* The structure of the curriculum then means that only the most able are exposed to anything like a formal proof activity and this becomes a self-fulfilling prophecy. An important task is to widen the notion of proof and make connections across the mathematics curriculum at *all* levels. A possibly useful definition of a "real" proof is "a convincing communication that answers the question why".

It is certainly the case that we, as teachers, often explain the "why" by illustrating with a few examples, or with a picture. Yet we are surprised, perhaps *alarmed,* when pupils do the same. As one mathematician has said, generalisations only work in *particular* cases. In making the transition from informal to formal notion of proof we perhaps need to focus on the introduction of systematic language and the progressive use of symbols.

Assessing the range of reasoning ability in geometry

This involves examining the questions
What? (content, processes, reasoning, ..)
How? (Piaget, van Hiete, SOLO, ...)
Why? (examination, planning, research, control, ...)
Who?

Tests that purport to assess the range of reasoning ability in geometry have been developed using the van Hiele model. These can be easy to administer and easy to score (which can make them attractive). The tests can also be used internationally, and can, for may students, (it is claimed) effectively identify the van Hiele level. However, these tests can not assign a level to a student who is in transition between levels and, it is reported, the tests are not very reliable. What is more, the theoretical basis of the van Hiele model is also being called into question. There are some who refer to the van Hiele model *only* as a good example of how easy it is to confuse a number of crucial issues. For example, they claim that level of 'thinking' is *not* the same thing as level of knowledge and that, in addition, for instance, visualisation is *not* the same at each level.

The SOLO (structure of observed learning outcomes) taxonomy attempts to give some idea of student understanding *and* some indication for the teacher of the way forward. It can be seen to incorporate van Hiele but is even more ambitious. It could be argued that, as it is based on current curricula practice and is enormously complex, it may be of limited practical value in assisting in the long, complex problem of interpreting observed behaviour.

Final remarks

The above gives only a glimpse of the quality and quantity of discussion at the ICMI conference. This report can not address the discussions that took place in the other working groups and there is insufficient space to do justice to the round table presentations. There was general agreement at the conference that more *research* is necessary in order to effectively address the wide range of issues that were discussed. The resulting book, due in 1997, will undoubtedly contain useful perspectives on the teaching of geometry for the 21 st century and will provide some guidance for the research endeavour. There remains a great deal of work to be done.

References

Bishop, A.J. (1983). 'Space and Geometry'. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematical Concepts and Processes.* (pp. 173-203). Orlando, FICl: Academic Press.

Mammana, C (Ed) (1995), *PerSPectives on the Teaching of Geometry for the 21st Century: proceedings of the JCMl study conference.* Catania, It: University of Catania.