Design Decisions: A Microworld for Mathematical Generalisation

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This paper provides the preliminary analysis of a study in which year 7 students interacted with *eXpresser*; a microworld designed to support students' transition from the 'specific' to the 'general' by constructing figural patterns of square tiles and finding rules to describe their model constructions. We present evidence that support three key design decisions of eXpresser and discuss how these features facilitate students' expression of generalisation.

Introduction

Utilising young students' natural algebraic ideas and developing them through carefully designed digital media appears to be a fruitful avenue to explore in teaching of algebra. Students can verbalise algebraic rules in natural language but struggle to use mathematical language (Warren and Cooper 2008). They often fail to see the rationale, let alone the power, of generalisation. The MiGen project[4] is tackling this problem by supporting 11-14 year old students in their problem-solving during generalisation tasks, and providing them with a rationale for finding and checking general constructions and rules. The core of the MiGen system is a microworld, named *eXpresser*, in which students can build figural patterns of square coloured tiles and express the rules underlying them. The eXpresser seeks to provide students with a model for generalisation that could be used as a precursor to introducing algebra, to help them develop an algebraic 'habit of mind' (Cuoco et al. 1997). In this paper, we will focus on student interactions with specific functionalities of expresser arising from three of the key design decisions (all of the design decisions are extensively described in Noss et al. 2009, Geraniou et al. 2009).

The Microworld, the eXpresser

This section provides a short description of expresser (for a detailed description the reader is referred to Noss et al. 2009, Geraniou et al. 2009). Students are presented with tasks such as the one shown in Figure 1. The pattern is animated and the figure number changes accordingly. Students then build constructions for the patterns by expressing what they 'see' as the structure of the pattern, making explicit any of their rules, and finally using the relationships to obtain the number of tiles needed in the pattern using the metaphor of colouring the right number.

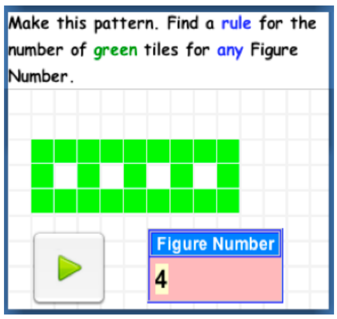

Figure 1. Train-track activity

Figure 2 shows a snapshot of the microworld. The student has just finished constructing the given pattern using a C-shaped building block (shown in A). She associated the Figure Number of the task with the number of 'holes' in the pattern and used a variable with this name. To create the pattern, the building block is repeated as many times as the value of the variable 'holes' (B), in this case 3. In

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every repetition the block is placed two squares across (C) and zero places down (D). To complete the pattern the student needed a 'line' of three tiles at the end.

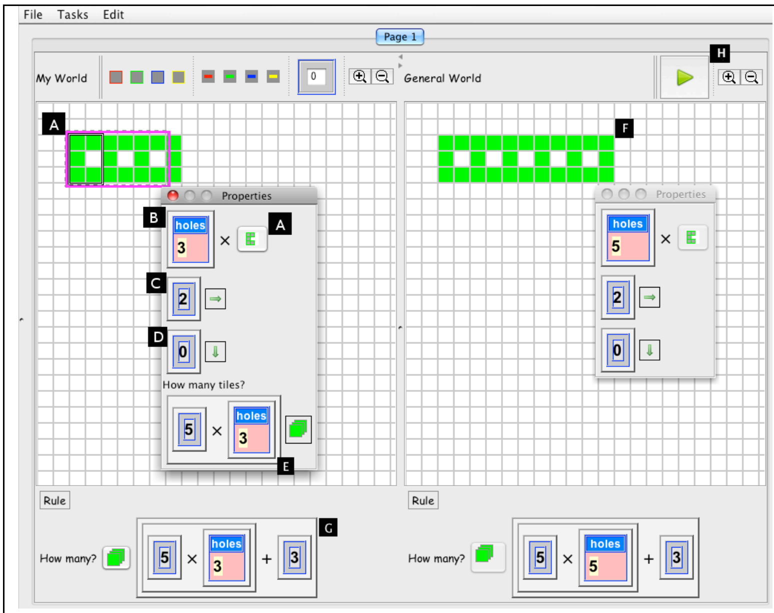

Figure 2. Constructing and describing with rules a pattern in the eXpresser. Letters highlight the main features: A) base shape to be repeated to make a component of the pattern, B) number of repetitions (in this case the value of the variable `holes') C) Number of squares to move to the right after each repetition (in this case 2) D) Number of squares to move down after each repetition E) Units of colours required to paint the pattern component F) Any variable used in the construction takes a random value in the 'General World' (G) A rule for the total number of units of colours required to paint the whole pattern in a general way. (H) Patterns can be animated when the system changes the value of the variables and are coloured if the rule is correct.

Finally the student has to colour the pattern by allocating to it the exact number of coloured tiles. In the case of the sub-pattern made of C-shapes, this required 15 tiles, i.e. 5 * holes (E).

As students build their constructions in 'My World', a 'General World' can be seen alongside it (right window). This world exactly mirrors 'My World' until the student has unlocked a number (i.e. created a variable), at which point eXpresser randomly changes its value in the 'General World'. The idea of 'locked' and 'unlocked' numbers was introduced to allow students to specify whether a number should stay the same (locked, i.e. constants) or could change (unlocked, i.e. variables). In 'My World' students can edit the unlocked numbers whereas in the 'General World', eXpresser chooses random values for the 'unlocked' numbers. For example, in the snapshot the value of `holes' is 5 resulting in a different instance of the pattern (F). The pattern in the 'General World' is coloured only when students express correct *general* rules for the pattern. Students cannot interact with the 'General World' except by clicking the *play* button (H) to animate their patterns and test its generality.

Methodology

Throughout the development of eXpresser, we have followed an iterative design process, interleaving software development phases with pilot studies with students of our target age (11-14 years old). We have also integrated feedback from teachers and teacher educators as well as the students who participated in the studies. The study presented here comprised three activities: the first introduced the functionalities of the microworld through a series of eight video-tutorials, the second where students used eXpresser to build a simple pattern, and the third, called Train-track (as presented in Figure 1) in which students were asked to find a rule that gave the number of green tiles for any figure number. Sixteen students who participated in this study were asked questions throughout their interactions designed to reveal their comprehension of the system and the particular design features under study.

Students' Interactions with eXpresser

This section focuses on three key design features and student reactions to them.

Animated task presentation to provide a rationale for generality

All patterns were presented to students animated changing at fixed time intervals showing a different instance of the pattern each time. This made it hard for students to count the number of green tiles (e.g. Figure 1), while allowing them to see the variant and invariant parts of the pattern. This presentation provided a rationale for deriving a rule that output the number of green tiles for *any* instance of the pattern, i.e. a 'general' rule giving concrete instantiation to the meaning of 'any'. In addition, they were given the chance to see their constructions animated by clicking the play button in the General World (Figure 2H) that allowed them to validate their model's generality.

One student when asked to describe the way she saw the task in Figure 1 replied:

> Ann: 'These are flashing green squares and it [referring to the figure number] changes number'.

At the end of the session, she was asked: Why do you think we presented the task like this (animated)?

> Ann: because you can't make it move on paper. […] it's not just one number and it doesn't stay the same.

Besides the dynamic potential the technology offers in comparison to conventional paper presentation, Ann seemed to have gained a more 'general' perspective after interacting with eXpresser. Ann expressed the notion of a variable in her own words as 'it doesn't stay the same'. We could also see how she switched to paying attention to the value of the number of tiles rather than the changing shape – a rather clear instance of differentiating (or at least seeing the importance) between the object and its value- and in so doing sees the connection between the changing pattern and the figure number.

Working on a specific case 'with an eye' on the general

The most crucial, yet difficult, step for students is to distinguish between what stays the same and what changes between different instances of a pattern. Students tend to work with a particular example and struggle to find the solution for an infinite number of invisible and unspecified cases. We therefore designed the system with two separate, yet linked, windows as described earlier. If students make two or more sub-patterns (e.g. see figure 3) to build their pattern, they *have* to express the necessary relationship(s) between the unlocked numbers if their construction is not to be messed-up in the General World (random numbers are chosen for each unlocked number). This happened with Nancy who made the train-track pattern using two sub-patterns of vertical lines and vertical with a gap but any relationship between them. She was surprised when she noticed the messed-up pattern in the General World, but this helped her realise that she had to express that the number of repetitions of the building block A1 as one more than the number of the building block B1.

> Researcher: what happened there [pointing at the General World]?
>
> Nancy: It is not joined up ... because you only have 1,2,3 of them and it's got to be , one more extra. [...] If there is 1,2,3,4,5 of these, that means that you always have to add one more. You might have 5 then you need to add an extra line, so that's 6.

By using the word '*always*', Nancy revealed her ability to generalise in natural language and eXpresser allowed her to express this relationship by using locked and unlocked numbers. She was then able to construct the general rule for her model (as mentioned later in this section).

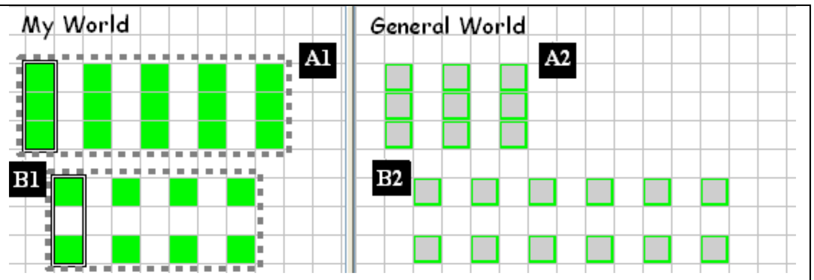

Figure 3. Nancy's construction from two sub-patterns**.** The first (A1) is made by repeating a building block of 3 tiles five times. Since the number of repetitions was unlocked, it was changed randomly to 3 in the General World (A2). The second sub-pattern (B1) is made by repeating a block of 2 tiles four times. In the General World (B2) the unlocked number was changed to 6. This helped her realise that she had to express the number of repetitions of the building block A1 in terms of the number of repetitions of B1.

A powerful description of the General World and its overall purpose was given by Kathy:

> Kathy: in My World is like a plan and in the General World is what comes to life and actually moves. It [General World] makes it live and animates it. […] The only difference is that on My World you put the figure number and it doesn't change, whereas in the General World, the figure number changes.

Kathy succeeded in seeing the rationale of the General World as a window that provided different instances of her pattern and therefore a window on to her own generalisations. She also had a rather powerful metaphor of a 'plan' for the construction of a rule via the specific case, one which could be 'implemented' (our term) in the general case.

Mutually supportive model construction and rule construction

In eXpresser, colouring a pattern requires an expression (a rule) that allocates the correct number of tiles in all cases. Based on the constructed pattern, students need to find the exact number of coloured tiles for their pattern. For local patterns (figure 2E) this can be expressed as a multiplication of the number of repetitions and the number of tiles in the building block. We were convinced that this design decision would give a direct experience with the number of tiles needed to construct the building block and encourage students to look at the structure of their pattern. It would further support them to construct a rule that calculates the number of coloured tiles using the number of tiles of the building block as a coefficient. For their complete construction and colouring of the pattern, they need to provide a combination of local rules.

Nancy, for example, when constructing the model in Figure 3, derived two rules for her two sub-patterns. Once she had coloured the individual patterns, it only required a small step to understand that the general rule can be constructed by adding the two together (see Figure 4). It seemed, therefore, that for most students the process of model construction supported the construction of a final rule for the pattern. In addition, when asked to show how they worked out their rule, they usually employ elements of their model to describe it (e.g. see figures 5 and 6).

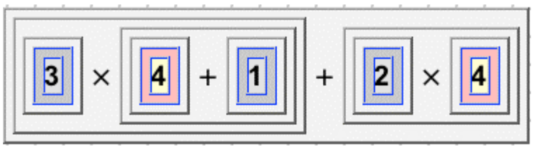

Figure 4. Nancy's rule

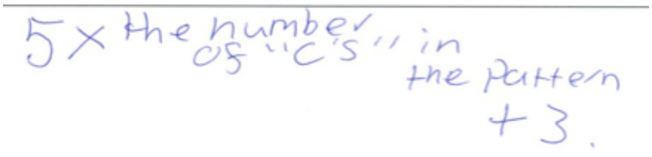

Figure 5. Henry's rule for train-track task

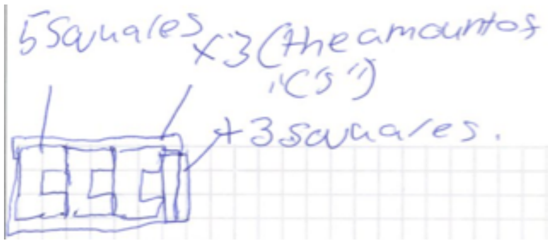

Figure 6. Henry's demonstration of his rule based on his model.

Conclusion

Our attempt to build a system that gives students the opportunity to work with the particular and general at the same time is still ongoing. The approach we chose through the three key design ideas is not the only existing one, but has advantages compared to the conventional methods with paper-and-pencil. We have therefore some provisional evidence interaction with eXpresser as a model of generalisation engages students, provokes them to think about generalisable structures and helps them to make the transition from numbers to variables in a way that is meaningful. In the interviews, all students used their rules to give the right number of green tiles for different figure numbers using structural reasoning and not pattern spotting. Kathy, for example, having found a similar rule to Henry, said:

Kathy:	my rule was 5 green times the figure number add 3.
Researcher:	so for figure number 6, what would it be? How many would there be?
Kathy:	it would be… 5 times 6…30…it would be 33.
Researcher:	for 12?
Kathy:	it would be 5 times 12…60…add 3 ….63.
Researcher:	for 600?
Kathy:	5 times 6…300…and then 3…

It seems that interaction with eXpresser discouraged students from calculating and spotting patterns at the expense of expressing structure (see, for example, with reference to this type of task, Noss et al. 1997, Healy and Hoyles 2000, Küchemann and Hoyles 2009). Paper and pencil approaches tend to lead to the referents of the relevant variables becoming obscured, thus limiting students' propensity to conceptualise relationships between variables, to justify and use them in a meaningful way. With eXpresser, students have to construct models employing the structures they see and find rules with generality in mind.

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