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These proceedings consist of short research reports which were written for the BSRLM day conference on 28 February 2009. The aim of the proceedings is to communicate to the research community the collective research represented at BSRLM conferences, as quickly as possible.

We hope that members will use the proceedings to give feedback to the authors and that through discussion and debate we will develop an energetic and critical research community. We particularly welcome presentations and papers from new researchers.

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Informal Proceedings of the British Society for Research into Learning Mathematics (BSRLM)

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This paper explores the behaviour, attitudes and beliefs of primary school pupils towards mathematics in the classroom and the impact that this may have on their mathematical ability. The study focused on year 3 pupils from a local school, some of whom took part in focus groups towards the end of the project. The children completed short worksheets, which were used to stimulate a guided discussion on what aspects of mathematics the children liked and disliked. The aim of this project was to isolate possible causes of negative attitudes towards mathematics and to discuss what their implications might be.

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Jenni Back and Marie Joubert

University of Plymouth and University of Bristol

This paper presents some of the findings of the Researching Effective Continuing professional development in Mathematics Education (RECME) Project which was set up to investigate, amongst other things, the role of research in 'effective' CPD for teachers of mathematics. The focus in this paper is on a CPD initiative that involved a network of teachers and early years practitioners. The Early Years Foundation Stage (EYFS) covers the care and education of children from birth to five years old and the place of mathematics in these settings has historically been problematic (Gifford 2005; Griffiths 1994; Moyles 1994); we suggest this makes this initiative particularly interesting. During meetings, which involved practitioners from a variety of settings, participants carefully considered children's mathematical work, especially their spontaneous mathematical graphics (Worthington & Carruthers 2003). This focus led the practitioners to consider ways in which they might support the children's mathematical development in EYFS settings. We suggest that the professional development of the participants occurred through this collaborative work on researching children's mathematics in the classroom.

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Bowker, A., Hennessy, S., Dawes, M. & Deaney, R.

University of Cambridge

The T-MEDIA[1] research project produced an interactive CD-ROM containing a video-based case study of teaching and learning with technology (graphing software, spreadsheet and online games using data projector and laptops) in one secondary mathematics classroom. Designed as a tool for teacher-led, collaborative professional development, the resource aims to stimulate debate rather than present a model of best practice. In the follow-up project outlined here, groups of teachers in 3 schools discussed the pedagogical approaches portrayed, planned a lesson in response, observed each other and reflected together on the outcomes and implications for practice. We present the outcomes of these trials and our development of a 'toolkit' that might guide other departments' use of the resource for professional development.

[1] 'Exploring Teacher Mediation of Subject Learning with ICT: A Multimedia Approach' (2005-2007). Funded by the UK Economic and Social Research Council (RES000230825).

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Ayshea J. Craig

Institute of Education, University of London

This paper argues for the importance of re-examining theoretical assumptions in research into mental calculation strategies and strategic thinking in mathematics education. By contrasting research into strategic thinking in mathematics education with that in cognitive and developmental psychology, three areas are identified where important details of the model of strategic thinking are left unexplored in education research while being dealt with more thoroughly in the psychological literature. The areas identified are: the positing of innate processes; the nature of memory; and the relation between conscious and unconscious mental processes. The status and reliability of introspective reports on mental processes are discussed as an illustration of the potential of research in psychology to further inform mathematics education research in this area.

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Thérèse Dooley

University of Cambridge, U.K. and St. Patrick’s College, Dublin

Although plenary sessions are common to mathematics lessons, they are often characterized by traditional approaches that endorse the position of mathematics as a kind of received knowledge and the teacher as sole validator of students’ input. A socio-constructivist view of mathematics calls for a more conversational style of interaction among participants. In this paper an account will be given of a lesson in which children aged 9 – 10 years calculated the sum of integers from one to one hundred. Particular attention will be paid to one pupil, Anne, and her reassessment of a conjecture that she made early in the lesson. I suggest that particular teacher ‘moves’ facilitated engagement of other students with her idea and that this was one factor that led to her new insight.

Julie-Ann Edwards

University of Southampton, School of Education

This paper uses socio-constructivist and socio-cultural lenses to examine transcripts of pupils' peer talk recorded while they were undertaking open-ended mathematical tasks in a naturalistic classroom setting. I discuss the two theoretical frames and then present episodes of peer talk from pupils between 12 and 14 years old which demonstrate how a socio-constructivist view of the zone of proximal development is enacted, and how a socio-cultural lens offers a window on social aspects of these established working groups which serve to provide the necessary support to enable all members of the group to access the mathematical knowledge being constructed.

Jeremy Hodgen[a*], Dietmar Küchemann[a], Margaret Brown[a] & Robert Coe[b]

[a]*King's College London,* [b]*University of Durham*

In this paper we present some preliminary data from the ESRC funded ICCAMS project, and compare current Key Stage 3 students' performance on fractions and decimals items with students from 1977. We also present some interview data concerning students' models of fractions, and in particular their use of diagrams to represent part-whole relationships.

Allison Lu Yu-Wen

Queens' College, Faculty of Education, University of Cambridge, UK

The idea of the integration of dynamic geometry and computer algebra and the use of open-source software in mathematics teaching underpins new approaches to studying teachers' thinking and technological artefacts in use. This study opens by reviewing the evolving design of dynamic geometry and computer algebra; teachers' conceptions and pioneering uses of GeoGebra; and early sketches of GeoGebra mainstream use in teaching practices. This research has investigated English and Taiwanese upper-secondary teachers' attitudes and practices regarding GeoGebra. More specifically, it has sought to gain an understanding of the teachers' conceptions of technology and how their pedagogies incorporate dynamic manipulation with GeoGebra into mathematical discourse.

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Peter Osmon

Department of Education and Professional Studies, King's College London

Meetings with concerned groups of academics with a particular interest in the mathematics knowledge of students when they arrive at university are reported. There was general agreement on two immediately tractable issues and the appropriate actions: an A-level mathematics curriculum without options, so as to maximize students' knowledge in common, and examinations that test understanding and not merely memory and manipulative skill- so as to encourage deeper learning than at present. The relatively low numbers taking A-level mathematics is a much tougher issue. The consequences include many university courses in quantitative subjects admitting students without A-level mathematics, and adapting content and teaching accordingly, so as to survive. The underlying problem is to understand the unpopularity of mathematics after GCSE and what might be done about it.

The Validation of a Semantic Model for the Interpretation of Mathematics in an Applied Mathematics Problem **73**

Michael Peters[a] and Ted Graham[b]

[a]Learning Development Centre, Aston University; [b]Centre for Teaching Mathematics, University of Plymouth.

The semantic model proposed by Peters (2008) was developed whilst working with learners of mathematics solving algebraic problems. In order to investigate in more detail the role of the parsing process and its relationship to the lexicon, a different set of questions were devised based on Laurillard's (2002) work with undergraduate students. These same questions were also given to a set of mathematics tutors so that a comparison could be made between the two groups and to see if their behaviour could be explained using the semantic model. The analysis of these sets of data indeed show the importance of the parsing process and as predicted by the model, a competent mathematician employs a top-down parsing strategy.

responses are discussed. Data from three case studies suggest that the framework helped participants to consider their unplanned actions when teaching mathematics. There was also evidence that over the course of the study the participants become more able to act contingently in relation to the mathematical content of their teaching.

Working group report

BSRLM Geometry working group: Establishing a professional development network to support teachers using dynamic mathematics software *GeoGebra* **97**

Keith Jones, Zsolt Lavicza, Markus Hohenwarter, Allison Lu, Mark Dawes, Alison Parish, Michael Borcherds

University of Southampton, UK; University of Cambridge, UK; Florida State University, USA; Comberton Village College, UK; University of Warwick, UK; Queen Mary's Grammar School, UK

The embedding of technology into mathematics teaching is known to be a complex process. *GeoGebra*, an open-source dynamic mathematics software that incorporates geometry and algebra into a single package, is proving popular with teachers - yet solely having access to such technology can be insufficient for the successful integration of technology into teaching. This paper reports on aspects of an NCETM-funded project that involved nine experienced teachers collaborating in developing ways of providing professional development and support for other teachers across England in the use of *GeoGebra* in teaching mathematics. The participating teachers tried various approaches to better integrate the use of *GeoGebra* into the mathematics curriculum (especially in geometry) and they designed and led professional development workshops for other teachers. As a result, the project initiated a core group which has started to be a source of support and professional development for other teachers of mathematics in the use of *GeoGebra*.

Diagrams as interaction: The interpersonal (meta)function of geometrical diagrams

Jehad Alshwaikh, Institute of Education, University of London

Diagrams are part an parcel of mathematics. However, the main stream among mathematician is prejudiced against the use of diagrams in public. In my PhD study, I consider diagrams as a semiotic mode of representation and communication which enable us to construct mathematical meaning. I suggest a descriptive 'trifunctional' framework that can be used as a tool to analyse the kinds of meanings afforded by diagrams in mathematical discourse. In this paper, only the interpersonal function of the diagrammatic mode is considered with illustrations. In specific, I consider labels, neat-rough diagrams and modality as realisations of that function. Concluding remarks with challenges are presented at the end of the paper.

Keywords: Diagrams, mathematical discourse, multimodality social semiotics, representation and communication, interpersonal function.

Introduction:

Communication is inevitably multimodal where different modes, such as visual representations, gestures and actions, are used to convey meaning (e.g. Lemke 1999; Kress and van Leeuwen 2006; O'Halloran 1999; Morgan 1996). In that sense, mathematics is a multimodal discourse where different modes of representation and communication are used such as verbal language, algebraic notations, visual forms and gestures. These different modes have different meaning potentials, they contribute to the construction of meaning and the deployment of them carries the ‘unified’ meanings (Lemke 1999; Kress and van Leeuwen 2006). For example, the verbal language in mathematical texts has limited ability ‘to represent spatial relations such as the angles of a triangle (..) or irrational ratios’ (Lemke 1999). Thus we need diagrams or algebraic notations to represent these qualities or quantities.

The aim of my study is to develop a descriptive framework that can be used as a tool to analyse the role of diagrams in mathematical discourse adopting multimodal social semiotics. Halliday (1985) argues that any text fulfils three functions: ideational, the representation of our experiences in the world (e.g. the mathematical activity); interpersonal, the social relation constructed with the reader of the text; and textual, presenting the ideational and the interpersonal into a coherent text. Kress and van Leeuwen (2006) extended Halliday’s account and suggested a framework to read images and presented the notion of multimodality to express the different modes of representation and communication.

In mathematics education, a number of studies have adopted Halliday’s Social Semiotics approach, to look at different modes of communication and representation such as verbal language (Morgan 2006 has proposed a linguistic framework to describe the verbal mode of mathematical texts; 1996) and graphs and symbolism (O'Halloran 1999). Both Morgan and O'Halloran agree that, still, there is a room to investigate other modes of representation and communication of mathematical discourse.

The status of diagrams in mathematical discourse:

Mathematical diagrams are part and parcel of mathematics. They were used in ancient civilisations such as Old Babylon four thousand years ago (Robson 2008) and were an essential part of Greek mathematics (Netz 1999). Moreover, there is nearly consensus that diagrams are important in doing, learning and teaching mathematics mainly in visualisation, mathematical thinking and problem solving. However, the current mainstream among mathematicians is prejudiced against the use of diagrams or, more precisely, mathematicians 'deny' and hide the use of diagrams in their work (Dreyfus 1991). Mann (2007, 137) also states:

> When a mathematician explores new ideas or explains concepts to others, diagrams are useful, even essential. When she instead wishes to formally prove a theorem, diagrams must be swept to the side.

The main argument against the use of diagrams is that diagrams (or visual representations in general) are a) limited in representing knowledge with possible misuse of diagrams (Shin 1994); b) of an 'informal and personal nature' (Misfeldt 2007) and c) unreliable and lack rigour (Kulpa 2008). One main reason for this view is that the main stream thinking among mathematicians conceives mathematics as abstract, formal, impersonal and symbolic (Morgan 1996).

In my study, however, I consider diagrams as available resources for meaning-making and as a means for representation and communication for students to communicate with each other or with themselves in order to convey specific meanings. I suggest an analytic framework that can be used as a tool to analyse the kinds of meanings afforded by diagrams in mathematical discourse focusing on geometry. This trifunctional framework offers three interrelated different ways to look at diagrams as a semiotic resource: ideational, interpersonal and textual. In this paper I consider only the interpersonal function of diagrams because of the space available (for the ideational function see Alshwaikh 2008).

Diagrams as representation and communication: the Interpersonal Function

In the act of representation and communication the author produces an image, for example, to convey a meaning. While doing so, s/he creates a type of imaginary social relation with the viewer. Following Kress and van Leeuwen (2006), this relation is realised by contact, (social) distance, and modality.

Contact:

In his social semiotic account, Halliday (1985) (Kress and van Leeuwen 2006 follow him) distinguishes between two types of contact between the author of a text and the reader/viewer; demand and offer. Either the author demands 'something' from the viewer, for example to answer a question. Or the author offers 'something' to the viewer and in scientific texts the offer is, mostly, information. One main feature in geometry context I consider to contribute to this kind of relation between the author and the viewer in geometrical diagrams is labelling.

Labelling

In geometry, labels are given to the components of shapes or diagrams: the vertices, the sides, the angles and parts of the diagram. Labels are either of offer-labels type or demand-labels type. [i] (Indeed there diagrams where the two types are combined.)

Offer-labels: This type offers information about geometrical diagrams and does not ask any action to be taken. It expresses either a) geometrical relationships such as equality, parallelism (Figure 1) or b) specific quantities (Figure 2).

In Figure 1, all labels are presented to show properties and geometrical relationships in diagrams. The general-type of these labels suggests that they are used to introduce definitions or qualities of these diagrams. This practice often occurs in school textbooks. In other words, presenting labels in a general form suggests an authority (a mathematical one) who says what the definition of, for example, a parallelogram is.

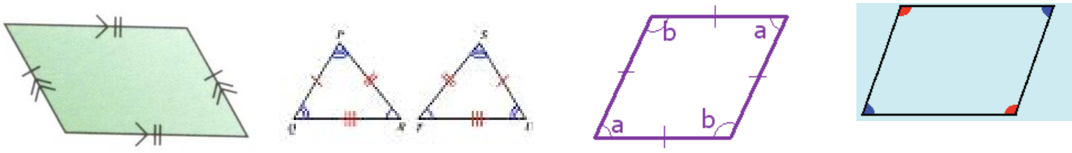


Figure 1: Offer-labels express geometrical relationships: same label (and/or colour) means equality or parallelism

On the other hand, labels in Figure 2 show specific examples with specific quantities. In mathematical discourse, this type suggests 'less' authority for the author than the previous type (general-type labels) because of the current mathematical mainstream understanding which values the general, abstract and formal prepositions, such as general properties and definitions (e.g. Davis and Hersh 1981), higher than the specific examples. Moreover, solutions to specific problems indicate that they were produced by someone with lower authority in response to a problem posed by higher authority. Thus one possible interpretation for diagrams in Figure 2 is that these are examples to illustrate the general case, 'a rectangular trapezium with bases 16 and 24 meters long and a height 10 meters' or 'this is an example of a scalene triangle'. Another possibility is that a student drew these diagrams in order to solve specific problems and s/he is showing them to the teacher/assessor.

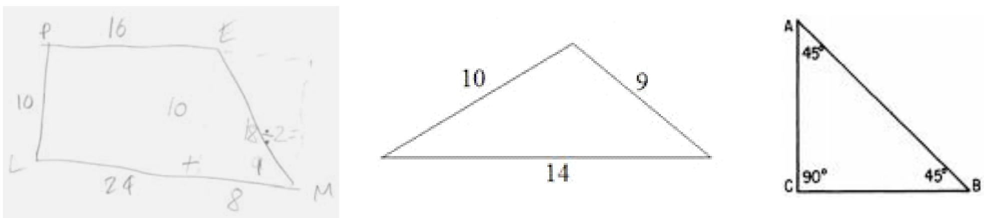
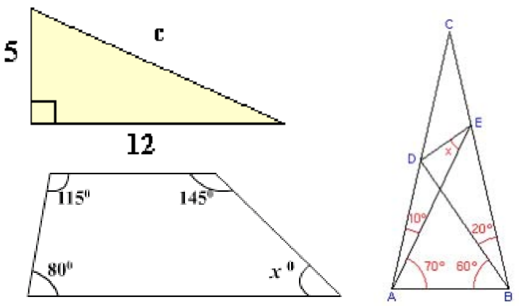


Figure 2: Offer-Labels express specific quantities



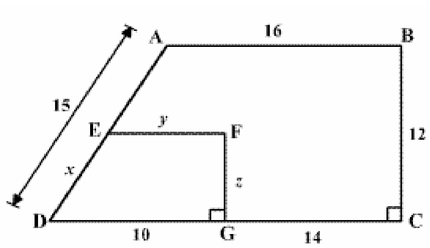


Figure 3: Demand-labels: unknown quantities

Figure 4: Demand-Labels: variable names

Demand-labels: This type of labels is realised by the presence of either a) unknown quantities (Figure 3) or b) variable names (Figure 4). It asks for a mathematical action to be done by the viewer which is to find the value of the quantity (finding the value of 'c' and 'x' in Figure 3, or the variable (x, y and z in Figure 4). In the context of school mathematics all these diagrams suggest an authority which asks a student to do something.[ii] Again, as in the offer-labels, the more general and abstract propositions the higher authority involved.

(Social) Distance

Kress and van Leeuwen (2006) consider the choices made between close-up, medium shot or long shot contribute to the meaning of image. In other words, the distance between the represented 'participants' in the image and the viewer of the image plays role in establishing a relation between them. Such physical distance is not realised in geometric diagrams. However, and following Morgan (1996), I consider that distance is expressed by the degree of 'neatness' of the diagram. In producing diagrams, the authors (mathematicians, teachers, students, etc.) draw accurate or rough diagram depending on the interest of the author, the context and the audience. A neat diagram 'indicates that the text is formal and that there is some distance in the relationship between the author and the reader' (Morgan 1996). On the other hand, a rough diagram suggests an intimate relation with the viewer or appears to be 'private' drawn while the producer works alone or for a personal use. Figures 5 and 6 show diagram drawn by students participated in my study to the same problems. As shown, they chose to present their diagrams differently.

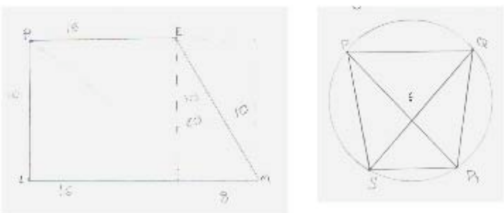


Figure 5: Neat diagrams

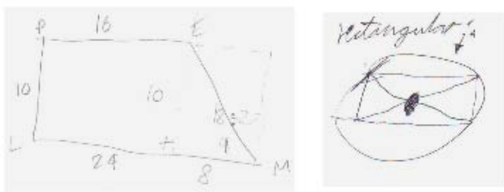


Figure 6: Rough diagrams

Modality

Modality in language refers to the degree of certainty and truth of statements about the world that is realised by the auxiliary verbs such as may, will and must and their adjectives such as possible, probable and certain (Kress and van Leeuwen 2006; Morgan 1996). In *Reading Images*, Kress and van Leeuwen distinguish between naturalistic modality and scientific modality depends on the social group which produces the image. The former refers to how the representation is 'close to' or 'true' in representing the reality and the photographs taken by camera are good examples. Scientific or abstract modality represents the reality in abstract mode such as geometric shapes or diagrams.

"Reality is in the eye of the beholder; or rather, what is regarded as real depends on how reality is defined by a particular social group" (Kress and van Leeuwen 2006). In mathematical discourse, 'more abstract approach is likely to be judged by teacher/assessor to demonstrate a higher level of mathematical thinking' (Morgan 1996). In general, it's not common to use naturalistic modality in (modern) mathematical texts, i.e. one rarely uses photograph or draw pictures to solve mathematical problem.[iii] Actually the dominant values and beliefs among mathematicians are that mathematics is abstract formal, impersonal and symbolic and school mathematics is not an exception. Hence, schematic or abstract diagrams are considered 'more' mathematical within the discourse of mathematics.

Clearly this issue is a social one which brings with it the issue of power as well. All the participant students in my study drew this type of diagrams. Drawing naturalistic or figurative diagrams to solve problems may consider by teachers/assessors as 'low level' of achievement or performance (Morgan 1996). One potential meaning arises when students draw these abstract diagrams is to announce their membership in the mathematical community or, at least, to say we know what mathematics is about. Other possible interpretation, however, is to challenge the teaching process and textbooks: to what extent do teachers and textbook present choices for students to present their own way of problem solving?

Concluding remarks and challenges:

Considering mathematics as a social and cultural practice, I argue that the use of diagrams is just as much an essential part of mathematical discourse as other modes, e.g. the linguistic and the symbolic. It is the practice of mathematicians that at some point turned to prejudice against the use of diagrams despite the fact that that use is and was essential. The suggested framework contributes to the analysis of mathematical discourse and practice in school mathematics (the way textbooks and teachers (re)present diagrams identifying the meaning potentials they carry) and how students make use of diagrams in their solutions.

However, studying the diagrammatic mode of representation and communication in mathematical discourse is not straightforward and does not lack challenges. I want to raise the challenges I face in interpreting the kind of social relations in offer- and demand-labels. Although I presented a general identification for the kind of social relation in these labels, there are different questions to think about:

- What kinds of social relations may labels offer/suggest? My point here is to ask whether students, for instance, would differently label their diagrams in solving problems to their teachers from their peers.
- Do mathematicians use labels in different ways from students? In other words, do mathematicians label their diagrams in a different way if they work on their own, with their colleagues or with their students?

These questions raise the issue of context in which diagrams are produced and used. I have to say that my study concerns about school geometry practice and that all diagrams I am using are drawn from within that context. In other words, any potential interpretations of the social function are dependent on that particular social practice. Furthermore, not only may individuals use labels (or other specific features) differently when engaged in different social practices but the ‘same’ feature may be interpreted differently in different contexts.

Notes:

[i] There is another type of labelling, *naming*, in which names are given to vertices, sides and parts of the diagram such as A, B, X, Y, etc. This type is different from other labels since neither information is offered nor actions demanded. It may be, however, considered as a reference to the viewer to refer to while 'reading' or solving a problem, and in that sense I consider it in the textual meaning.

[ii] This is also the case in an academic research article where the author may be presenting an as yet unsolved problem as an admission that their research is incomplete. This raises the issue of the context of production of diagrams.

[iii] There are some exceptions to this, especially when modelling is involved, for example, a photo of a ball being thrown used in the process of mathematising projectiles. Also there are 'incidental' photos/pictures in school textbooks– see Dowling's (1996) discussion of how different kinds of pictures may construct different readers. However, these illustrations are not considered geometrical diagrams and hence fall outside the scope of the suggested framework.

Acknowledgment: I would like to thank Candia Morgan for her vital comments on earlier version of this paper.

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Exploring Children's Attitudes towards Mathematics

Ben Ashby

University of Warwick

This paper explores the behaviour, attitudes and beliefs of primary school pupils towards mathematics in the classroom and the impact that this may have on their mathematical ability. The study focused on year 3 pupils from a local school, some of whom took part in focus groups towards the end of the project. The children completed short worksheets, which were used to stimulate a guided discussion on what aspects of mathematics the children liked and disliked. The aim of this project was to isolate possible causes of negative attitudes towards mathematics and to discuss what their implications might be.

Keywords: Primary, Attitudes, Purpose, Anxiety, Confidence, Language, Reflection

Introduction

Mathematicians have long held a high level of respect amongst their academic peers. Yet the subject of mathematics, although revered, remains a source of anxiety and trepidation for a large number of people. Widespread negativity towards mathematics appears in many forms, from misrepresentation in the media to the social stigma that seems to surround those who are mathematically gifted. Children often set mathematics aside as a cause for concern, despite their limited exposure to it (Hoyles 1982). It is a subject unlike most others, since it requires a considerable amount of perseverance from the individual in order to succeed. A negative attitude towards mathematics could considerably reduce a person's willingness to persist with a problem. Without the ability to persevere, mathematical development is likely to be difficult. The purpose of this project is to determine the possible root causes of these negative attitudes towards mathematics.

The study focused on Year 3 pupils from a local school, some of whom took part in focus groups. Three focus groups were carried out, each consisting of four children with similar abilities. Children were selected based on observations from previous visits. Subjects were chosen if they displayed strong feelings for or against mathematics, or if they were at the extremes of the ability range. The focus groups lasted for approximately 30 minutes and were broken into two parts. Firstly, the children were given 10 minutes to attempt four questions tailored to their ability range. The questions involved symmetry, arithmetic, a word problem and a problem solving exercise. The remaining time was used to discuss what the children felt about mathematics, using the worksheet as a focal point.

It is hoped that this project will provide significant insights into why many children have a pessimistic outlook on mathematics and indicate where future research is needed.

Mathematics and its apparent lack of purpose

Children may find the nature of mathematics difficult to cope with as its wider reaching implications can be hard to see. Experiments are carried out for the physical sciences, pictures are drawn in art class and language skills are used in everyday interactions with other people. However, mathematics has a very formal written sense about it, where activities remain intangible to the child. From the remarks I witnessed in the focus groups, it seems that children find it difficult to make a connection between the work they do on paper and its practical applications. The following transcript is taken from the high-ability focus group:

Charlie: You need to be good with numeracy, say when you're say, shopping for something – You need to work out how much you're paying. You don't have to be a genius at it, but you have to be quite good at it.

Researcher: You mentioned shopping; do you think you use your numeracy skills outside of school?

Billy: Oh yes, I use it on...

Researcher: What sort of things? Daisy?

Billy: Counting out money.

Daisy: We don't really do numeracy outside; it's just to work on word problems, and multiplication and addition and subtraction, more than doing numeracy skills outside.

Charlie: Yeah, definitely. You couldn't basically live without knowing maths, because like when going shopping you need to know how much money you need. Also if you're going on holiday, you need to know how much money you're spending and stuff.

Similar remarks were made by children in the middle and low-ability focus groups:

(Middle-ability)

Faye: Yeah, maths is important, because it makes you more clever with adding things.

Gilly: Why's adding important?

Faye: Because then like you said you can count all your money and go to the shops and buy stuff.

(Low-ability)

Lisa: If you're a shopkeeper, and someone gave you like about £20, and something was like £15 and they didn't know much how much to give them back. And if you didn't know, you should learn more in your maths.

It was rather surprising to see pupils across the entire ability range unable to make connections between mathematics and its many practical uses. Counting money was the only association that they were able to make, even though it had not been covered in recent work. It is interesting that the high achievers, although mathematically gifted, could not establish any more real world applications than the low achievers. However, the low achievers present more of a concern, as motivation to improve their mathematical understanding cannot be aided by their innate ability. Certainly, the children cannot be expected to make these connections without assistance from a teacher. In fact, some believe that the most effective teachers are connectionists (Askew et al. 1997), although perhaps there is currently insufficient emphasis on the practical uses of mathematics in the curriculum.

Human nature does not favour futile endeavours; if a difficult task appears to have no purpose, then few will continue to follow it through. If low achievers are unable to see the wider benefits of having strong mathematical skills, then they may lack motivation, which is vital in a difficult subject such as mathematics.

Understanding the purpose of mathematics should not only help to improve motivation, but could help in the actual formulation of concepts. In 1991, Harel and Tall discussed the importance of what they called 'the necessity principle':

> This principle states that the subject matter has to be presented in such a way that learners can see its necessity. For if students do not see the rationale for an idea (e.g., a definition of an operation, or a symbolization for a concept), the idea would seem to them as being evoked arbitrarily; it does not become a concept of the students. (Harel and Tall, 1991 41)

They believed that a notion is more likely to be abstracted successfully if the learner can acknowledge the necessity of the concept. In the context of this project, the learner needs to be aware of the purpose behind their work. For young learners, understanding the practical uses of mathematics could be sufficient to both motivate them and allow the necessity principle to be satisfied.

Further research is required on this issue, as its scope may be greater than previously thought. As with all the findings in this project, the data was collected from a small sample group, and so it may be difficult to generalise to a larger population. However, based on the remarkable similarities between responses in this particular classroom and the general attitude towards mathematics in our society, I would suggest that the apparent lack of purpose in mathematics is a sentiment felt by many.

Self-belief and mathematical ability

Nothing was more evident during the focus groups than the lack of self-belief shown by many of the children. Low and middle achievers quickly resigned themselves to failure, without truly attempting all of the questions on their worksheet. There was a consistent association of mathematics with ‘cleverness’, as many of the children felt not only that numeracy was harder than literacy, but that to be clever you had to be good at numeracy. In effect the children were implying that someone who excels in literacy will not be perceived as being clever unless they can display a similar exemplary ability in numeracy. As a result, children who perceived themselves to be weak felt that they would be incapable of solving harder mathematical problems. A girl from the middle-ability group remarked:

> Faye: I’m just going to do a simple answer, which is probably wrong.

While some would say that any answer is better than no answer, Faye’s decision to give up and guess occurred before she had given any real consideration to the question. This example was typical of her low confidence in mathematics; an attitude which I believe greatly misrepresents her ability.

Many of the children showed signs of anxiety whilst attempting the worksheets, shuffling awkwardly in their seats, glancing at their peers with worried expressions and making negative comments about the difficulty of the current task. Previous research into anxiety and mathematics (Hoyles, 1982) indicates that a connection may lie between an individual’s perceived ability and their level of success. The absolute nature of mathematics, where there is normally only one right answer, could add considerably to a negative attitude towards mathematics.

Overall, girls expressed much lower confidence than boys, even among the high achievers. They frequently attributed success and failure to external factors, such as luck and the perceived difficulty of a question. In comparison, most boys recognised that success was due to their own ability, and that failure was caused by either a lack of effort or understanding on their part. Whilst this distinction was not absolute it did apply to the vast majority of pupils that took part in the focus groups. The difference in attitudes towards mathematics between genders has been researched in depth by many, notably Stipek and Gralinski (1991). Although girls and boys are roughly equal in the league tables at GCSE level, there is a remarkable difference in A-level and University uptake. It is quite possible that primary school experiences are alienating girls from the subject, to the detriment of their long term mathematical development. The reason for this is currently unclear and warrants further research, as early childhood experiences could be discouraging many gifted female mathematicians from advanced study.

Difficulties with the language of mathematics

A lack of confidence was not the only cause of problems in the focus groups. Many of the children from the middle and low-ability groups appeared to struggle with the language of mathematics, often expressing nonsensical ideas. The following question and transcripts are taken from the middle-group:

Put the numbers 1-4 in the circles below so that each side adds up to 9.

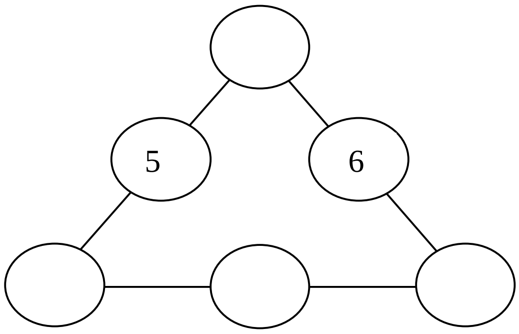

Figure 1 – Problem solving question for the middle group

Harry: 4 add 6 is 9; 4 add 5 is 9. So what's in the middle?

Harry has either forgotten the basic principles of addition and conservation, or he is getting confused with the language. Some children found it difficult to read a question and understand it at the same time, even though all the information that they required was written in front of them:

Faye: Does it have to add up to 9?

Researcher: Each side has to add up to 9.

Faye: Only 9?

Researcher: Yes.

Faye: So what's the biggest number you can go up to?

This was a common problem for Faye. Yet, if the question was read aloud to her, she was able to come up with the answer much faster than her peers. On reflection, she remarked:

Faye: That was only hard because I didn't understand it, but now I do understand it, it's sort of not hard.

Much of Faye's frustration seemed to stem from her failure to consistently and successfully understand questions that she was tasked with, and apply the correct procedures to solve them. This weakness somewhat masked her true ability, as she was clearly a stronger pupil than she first appeared.

Mathematics can appear as a foreign language to many people. It has its own alphabet, comprised of numbers and symbols, and is constructed with a complicated syntax. Just like a foreign language, it is easy to misinterpret. The children in the focus groups showed that they have significant problems translating this language into something useful that they can work with. For some children, it seems that the mathematical processes themselves are not problematic, it is instead a communication issue, and how they are able to interpret mathematical language.

Recent advances in affordable interactive technology for the classroom could go some way to easing this problem, as it enables mathematics to be communicated in new and interesting ways. However, examinations are almost entirely in written a format, which does not bode well for those who find written mathematics difficult to interpret. It would be interesting to see whether there is an increase in attainment when questions are read aloud.

The importance of reflection

The final point that I wish to discuss is the importance of reflection in the learning process. Time constraints were obvious during my visits to the school and several children commented about how they often felt rushed during numeracy lessons. We need to consider how this may hinder their mathematical development.

Educational researchers such as Beth, Piaget and Dubinsky developed ideas about constructivism as a theory of learning (Beth and Piaget 1966) and reflective abstraction (Dubinsky 1991), which may prove useful in determining the best course of action for today's young learners. The theory of reflective abstraction is aimed at older learners, who are engaged in advanced mathematical thinking, but we can take some lessons from this theory and apply it to primary education. It was suggested that reflective abstraction is a key step in understanding advanced mathematics. Before the learner can successfully use a concept, he must reflect and abstract properties from it in such a way that he can internally coordinate it. Then he can manipulate the concept for his own purposes. Older learners can be expected to conduct this process with little outside intervention, but the same cannot be said for primary school pupils. If they are to reflect in a similar way, then they need guidance from the teacher, which is extremely difficult in a rushed environment.

The plenary is effectively the reflection period in the three part lesson. However, many would argue that it is also the weakest section of the current lesson format. It comes at the end of the lesson, so children are more likely to lose interest and be distracted by thoughts about recess or the next activity. If this is the case, then a separate reflection period after the lesson may serve the children better. A separate reflection period would hopefully allow the children to challenge the ideas that they have recently learnt and overcome erroneous concepts. Encouraging reflection at an early age may even have long term benefits by improving children's general attitudes towards learning. Highlighting the links between topics may also help to enhance reflection, as well as creating a wider sense of purpose about mathematics.

Implications

For the researcher

This project has given several indications as to the causes of negative attitudes towards mathematics. The first priority should be to explore the link between children's perceptions of practical mathematical uses. This is an area that has seen little research in the past, but could provide significant insights into how to improve children's perceptions of mathematics. This problem is not restricted to any particular age group (it is displayed by many adults as well), so the scope of the research could be expanded across a much wider age range.

The second priority should be to investigate the differences in attitudes between genders. The difference was already apparent in this year 3 class; it would be interesting to see whether the roots could be traced back to earlier experiences in primary school, or influences from the home or media. Anxiety and difficulties with the language of mathematics are well known problems in the subject. However, it may be wise to reignite these points of discussion in the light of this project. It was clear from the focus groups that these two issues are causing more negativity towards mathematics than any others, and so should be taken very seriously. Deeper research into the origins of these issues could highlight ways to tackle not only the symptoms, but the causes too.

Finally, it may well be worth exploring what impact a separate reflection period would have on children's attitudes towards mathematics, as well as their ability. However, this project would be the hardest to plan and implement, due to restrictions on the school timetable and the impact that it may have on the children's learning.

For the teacher

Undoubtedly, the teacher faces an uphill struggle trying to balance a diverse range of abilities and attitudes, an ever changing curriculum and strict time constraints. However, there are several outcomes of this project that should be considered by the education community. For example, it may be worth exploring how the children perceive mathematics and its uses outside of school. By improving the understanding of the uses of mathematics, pupils will hopefully see the benefits of developing strong mathematical skills for more than just academic purposes. Likewise, low self-belief is an issue that all teachers can attempt to address. We need to dispel the notion that mathematics is a subject limited to geniuses and that children of all abilities can be successful in the subject.

The structure of the lesson and the time constraints of the school day should also be up for revision, as the current lesson format may not be the most efficient. The school curriculum is often subject to repetition, some of which may be avoidable with a subtle shift in lesson structure.

Conclusion

It is clear that children's attitudes towards mathematics can be influenced by a wide variety of factors. This project has gone some way to identifying what a few of these factors might be, but there is still plenty of scope for future research. In particular, children's views on practical uses of mathematics and the difference in attitudes between genders require further study. Additionally, the importance of reflection in primary education needs to be discussed in much greater detail.

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Reflecting on practice in early years' settings: developing teachers' understandings of children's early mathematics

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This paper presents some of the findings of the Researching Effective Continuing professional development in Mathematics Education (RECME) Project which was set up to investigate, amongst other things, the role of research in 'effective' CPD for teachers of mathematics. The focus in this paper is on a CPD initiative that involved a network of teachers and early years practitioners. The Early Years Foundation Stage (EYFS) covers the care and education of children from birth to five years old and the place of mathematics in these settings has historically been problematic (Gifford 2005; Griffiths 1994; Moyles 1994); we suggest this makes this initiative particularly interesting. During meetings, which involved practitioners from a variety of settings, participants carefully considered children's mathematical work, especially their spontaneous mathematical graphics (Worthington & Carruthers 2003). This focus led the practitioners to consider ways in which they might support the children's mathematical development in EYFS settings. We suggest that the professional development of the participants occurred through this collaborative work on researching children's mathematics in the classroom.

Early Years Foundation Stage, children's mathematics, mathematical graphics, professional development, classroom research

Introduction

The RECME project (Researching Effective Continuing Professional Development in Mathematics Education) was a major research project funded by the National Centre for Excellence in the Teaching of Mathematics (NCETM) which focused on analysing data gathered from researching thirty continuing professional development (CPD) initiatives in mathematics education from all phases of education from early years to further education. The RECME project involved a team of five researchers from a variety of research backgrounds and employed a range of research instruments which gave us a rich data set of qualitative and quantitative data. This account focuses on establishing the roles of research in professional development for teachers of mathematics. It outlines our findings and offers one case study initiative elaborating on the responses of one teacher to the professional development in which they were involved.

Amongst our key findings was a suggestion that teachers who gave accounts of changes in practice that they reported as sustained and profound had often been involved in their CPD in focusing on student learning and reflecting on the relationship between student learning and their own professional practice in the classroom. We also found that some of those teachers who were beginning to develop as leaders of CPD themselves were involved in reading and reflecting on educational research findings and it is this aspect on which this account focuses.

The case study initiative was set in the context of Early Years Foundation Stage education in which a group of teachers and practitioners established a network group in a city in the south west of England. Early Years Foundation Stage involves the education and care of young children from birth to five years old. The network of practitioners was initiated by two researchers, who seemed to be passionate about children's early mathematics, as evidenced by the fact that they ran courses and conferences in the area around a major city. The researchers, Melanie and Lizzy, said that their passion was illustrated by, amongst other things:

> … our strong desire to help make mathematics more meaningful, challenging, accessible and interesting for young children.
>
> … our deep and enduring interest in young children's learning.
>
> … our belief in the significance of research.

This had originated in their own teaching experiences as a result of careful research into teaching and learning mathematics and reflecting on their practice over a number of years.

Sarah went on a course run by the two researchers. At the course, Melanie and Lizzy suggested that forming groups at a grass-roots level would help to encourage and support teachers and other professionals in working with children's own mathematics. Sarah acted on this suggestion to form this group. As she said:

> I was enthused by Melanie and, having identified a gap in the curriculum in the transition between Foundation Stage (pre-school and reception) and Key Stage 1 (first years of formal schooling), which my own setting was seeking to fill, I was able to pursue the ideas.

Within the school, in which the case study teachers worked, over the previous year or so there had been a significant change in the approach to teaching in the reception year, especially in mathematics, which had been driven by new Early Years Foundation Stage Curriculum[1] guidance.

The leader, Sarah, described the work of the group as follows:

> I have introduced practice and understanding from the CPD I received from Penny and fed back to both my own setting and the group. It has made me research an area of the curriculum about which I am strangely passionate, reflect on my own understanding and practice, collect and collate evidence and share this with fellow maths enthusiasts within my school and the group.

So the group was set up partly in response to Sarah's perception of a gap in the curriculum and in provision for transition between Early Years Foundation Stage and Key Stage 1 in the school in which both teachers work, as well as in response to the suggestions made at the course Sarah attended.

[1] In the UK, the Early Years Foundation Stage (EYFS) was introduced in September 2008 and refers to the stage of education for children aged up to five years. In this report we use EYFS to refer to this phase. The revised curriculum for EYFS includes guidance on the kinds of learning opportunities that should be offered and the developmental progression that might be expected. In the context of EYFS settings children are offered many learning opportunities to engage in *problem solving, reasoning and numeracy*. However, unlike the *communication, language and literacy* section of the curriculum which emphasises the importance of mark-making, *problem solving, reasoning and numeracy* does not. The 'learner centred' approach of the Foundation stage curriculum contrasts with a more teacher directed approach that is sometimes characterised by heavily structured and teacher led approaches, for example worksheets and following a template to create artefacts.

The aims of the CPD, the intended professional development and change

Melanie and Sarah described the aims of the group as involving developing teachers' and other practitioners'[2] professional understanding of young children's understandings of mathematics and to supporting them in developing strategies to develop and support children's early mathematical development. They particularly focused on considering examples of children's work and reading relevant research literature related to this focus such as the work of the course leaders (Worthington and Carruthers 2003). This led them to observe children's spontaneous mathematics closely particularly in terms of mark making, problem solving and communication and to consider carefully the ways in which children make sense of mathematics in the early stages of recording it.

Melanie and Sarah hoped that the participants' practice would move away from imposing mathematics on children and work towards supporting children in developing their own mathematical understandings and representations in meaningful contexts. This in turn would support children in adopting conventional symbols, such as the numerals, by working with the children's own representations and understandings.

Content and processes of the CPD initiative

The group was informal and met about once every six weeks. It involved teachers and nursery nurses from a number of different primary and nursery schools in the local area. In most cases more than one teacher participated from each school. Meetings were held after school and the venue changed from school to school. They lasted an hour and a half, with refreshments provided by the host school. The leader of the group, Sarah, intended to delegate more responsibility for convening the meetings and managing the discussion to others in the group and welcomed participation at all levels from everyone. The group was observed to be supportive, open and egalitarian in its structures. For example, at the observed meeting Sarah did take the lead, but all the participants brought their own contributions and all commented freely on each others' observations without Sarah dominating the meeting. Towards the close of the meeting another member of the group offered to host the next meeting and the agenda for the following meeting was collaboratively decided upon as an outcome of the observed meeting. One of the participants said that colleagues from another school had seemed interested and the group decided that they should be included in the next meeting.

The group received no funding from the schools or any other source, except in the supply of venue and refreshments by the host school. The agenda and content of each session was decided upon co-operatively by the whole group, which meant each participant was supported in their participation by the relevance of the content of each session and the collegial support from their peers. During the observed meeting, the participants all contributed examples of children's spontaneous mathematical problem solving which they had observed in their own settings. These examples were shared with the group and the scenarios from which they had arisen were discussed. The topic had been chosen at the previous meeting in response to the focus of the Revised

[2] As well as teachers, a number of other groups of early years practitioners, who have professional training in child care and child development, work in EYFS settings including nursery nurses. A number of early years practitioners were participants in this group as well as some teachers.

Numeracy Strategy and in the Early Years Curriculum Guidance on problem solving. Sarah described this as follows:

> At the group meetings we share examples of our children's mathematical learning supported by photographs, quotes, samples of work etc. We are currently working towards a shared file of examples of children's problem solving as a resource for all members of the group. Sharing our experiences, children's work, information from Melanie, other CPD training and ideas, adds to our collective knowledge of teaching mathematics.

This sharing of children's work formed the substance of the observed meeting and included variety of examples which had been carefully analysed by the professional presenting it. In many cases, these examples involved accounts of what the children had done, examples of their productions in terms of marks made or artefacts created and photographs of the children in action. The group discussed in detail the mathematical aspects of each example and talked about how they could support the mathematical thinking that it represented.

Sarah's experiences

Sarah was an experienced teacher with a post graduate teaching qualification and had studied mathematics to GCSE level. She specialised in teaching in Early Years and Key Stage 1 and took a leading role within her school for provision within the reception year, and so managed the transition from EYFS to Key Stage 1.

Actual professional development

For Sarah the main gain from the group was in:

> … enabling me to continue to keep abreast of current thinking, be reflective and share my ideas and experiences with fellow early years practitioners, teachers and nursery nurses in the private and maintained schools, in a safe, supportive, non-threatening environment.

Her involvement in the group and attendance at various conferences in the area run by Melanie and Lizzy had developed her understanding and enthusiasm and she was in the process of becoming a researcher in her own classroom. This is evidenced by the following comment:

> I have done additional research to promote children's mathematical graphics and problem solving, which are the main things that the group has focused on so far.

Sarah had read research articles about children's mathematical graphics and problem solving and felt that this was important:

> I liked knowing that I am aware of current thinking, research and best practice.

Sarah's participation in the group had made a significant contribution to her professional development. She had become more confident, as she said:

> I feel more confident in my teaching of mathematics and proud of my school's early years team's development in this area. The group is a lot of extra work for me but I find it personally rewarding, professionally exciting and socially enjoyable. It has been great to visit other settings as we take it in turns to host the meetings – and the next meeting will be chaired by the person hosting the meeting, so I might feel less responsible!

For Sarah, one of the key outcomes of involvement was the opportunity to discuss ideas related to EYFS teaching with colleagues from within her school and other schools, and to share examples of good practice as well as current guidance and issues arising from practice.

Changes in practice

Sarah was now committed to practice focused on children's mathematics as a result of her extended study of children's mark making and problem solving. Sarah described this as a complete change from the worksheet- and textbook-based approach that used to exist in her school.

Evidence of this way of working was observable in Sarah's classroom. Examples of the children's spontaneous mathematical work were displayed in annotated form on the walls and in their books. The environment offered a range and variety of resources for mathematical investigation which were all freely available to the children. Sarah voiced her enthusiasm and passion for the CPD and her work in leading it but also expressed the sense of pressure that taking on a commitment to leadership of the group had engendered. She said she would like to:

> I would like to feel less pressured all the time so I could really get down and focus on the children, their understanding and interests and then work with them to develop their mathematics within a balanced and meaningful curriculum.

This quotation illustrates that for Sarah's belief that children's mathematical learning develops from the children's interests and she would like to spend more time considering their understanding and the meaning that they attach to their mathematical productions. Sarah voiced her frustration at finding it difficult to find time for this work that she regarded as important.

Student learning

Evidence of children's learning of mathematics was displayed on the notice boards around the classroom as well as in the children's books and they were able to articulate their mathematical understandings clearly. As Sarah said:

> The children in our classes have a positive attitude to sharing and representing their mathematical thinking. They are developing confidence in their mathematical graphics which are valued, they are developing fluency and a willingness to talk about their thinking. By focusing on problem solving they are identifying meaningful problems, rising to the challenge and developing a sense of achievement and satisfaction in finding a solution. They are sharing ideas and drawing on prior experiences to inform their strategies. Hopefully this positive attitude to mathematics and problem solving will stay with them. The children are able to demonstrate their individual ability and explore concepts beyond the normal curriculum.

The displayed work, both within children's books and on the walls, demonstrated the detailed observations and analysis that Sarah made of evidence of the children's mathematical thinking and understanding on a day-to-day basis. It also illustrated the importance of mathematics for the children in this class.

In a recent communication with the RECME team she said:

> It has been interesting that some practitioners have said that they feel it is "contrived" to encourage the children to make mathematical graphical representations so I suggested that they focus on WHY WE do it e.g. to help our thinking, to help us remember, to show someone else, to bring inside from the garden or share with another class, to take home etc. What was more interesting really was that the same practitioner felt perfectly comfortable getting her Nursery aged children to write speech bubbles.

This illustrates the role that Sarah had begun to take in supporting the professional development of her colleagues.

Conclusions

The changes that the teachers reported, including Sarah, made in their practice were fully in line with the aims of the CPD. We would suggest, from the evidence that we gathered from this initiative as well as others, that ways of working with teachers that facilitate their mutual support and offer them ownership of the content, purpose and direction of their CPD may be particularly effective in supporting radical changes in professional practice.

Participant ownership of this initiative helps to sustain involvement and that the members support one another in sustaining this passion and enthusiasm. Overall, the initiative supported the participants in their professional change by giving them a space for the detailed and joint consideration of children's mathematical thinking. It supported them in following up research sources that would support their analysis of the children's mathematical graphics and enabled them to encourage children to take charge of their own mathematical activity. It also offered them a supportive and encouraging arena in which their professional concerns and difficulties could be discussed.

Another significant feature of this initiative was its focus on careful consideration and analysis of children's mathematics, and the ways in which professionals can support and encourage the children's mathematical thinking and reasoning. We were struck by the emphasis on observing and analysing children's spontaneous mathematical activity. This emphasis seemed to shift the teachers' focus from teaching to learning and to give them the opportunity to consider the children's mathematical understanding and thinking. The teachers were then able to use this to support the children in their mathematical development and to plan appropriate adult-led activities that would help the children build their mathematical thinking and reasoning, such as the counting.

In conclusion this initiative involved opportunities to learn, experiment and reflect with colleagues about children's mathematical thinking and learning. It also exemplified compatibility between the ways of working with children and ways of working in the PD context in the sense that in both contexts the points of view of the participants were treated with respect and valued for their authenticity. There was a key place in the CPD of a deep engagement with processes of student learning and an element of respect for the professional practice of all those involved in the PD. It also involved the teacher as a researcher/observer in their classrooms and we suggest that this element may be very important in developing commitment to sustained change in professional practice as well as improved student learning through its focus on learning rather than teaching.

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Supporting professional development for ICT use in mathematics using the T-MEDIA multimedia resource

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The T-MEDIA[1] research project produced an interactive CD-ROM containing a video-based case study of teaching and learning with technology (graphing software, spreadsheet and online games using data projector and laptops) in one secondary mathematics classroom. Designed as a tool for teacher-led, collaborative professional development, the resource aims to stimulate debate rather than present a model of best practice. In the follow-up project outlined here, groups of teachers in 3 schools discussed the pedagogical approaches portrayed, planned a lesson in response, observed each other and reflected together on the outcomes and implications for practice. We present the outcomes of these trials and our development of a 'toolkit' that might guide other departments' use of the resource for professional development.

Keywords: professional development, ICT, video, secondary

Introduction

Schools in the UK have invested large amounts of their capital in data projection technology, which has generated an interest in how to support teachers to use these systems to enhance their teaching. Research into teacher learning suggests that stand-alone in-service workshops tend to be of limited value in developing sustained transformation in teacher practice (e,g. Muijs & Lindsay 2008). A more promising way forward appears to be a professional development initiative that *draws on practice and support within teachers' situated communities, encourages peer learning and reflective practice, and provides explicit opportunities for teachers to explore how they can introduce new ideas effectively into their classroom to promote student learning* (e.g. Clarke & Hollingsworth, 2002; Zwart, Wubbels, Bergen, & Bolhuis, 2007). Our work sought to incorporate these ideas into a programme of support for groups of teachers in their use of projection technology. The approach was informed by three phases of research activity: initial co-construction of theory between teachers and researchers (Hennessy & Deaney 2009, in press), development of a multimedia resource to stimulate teacher discussions, and the subsequent development of an accompanying 'toolkit' offering a framework for teachers' collaborative use of the resource. This paper reports on the development and trialling of the toolkit by three secondary mathematics departments and the reflections of teachers and an Advanced Skills Teacher-researcher (Dawes) on being involved with the project.

Teacher change

Our research draws on the literature on: teacher learning from a community of practice perspective, reflective practice and peer learning. While traditional off-site

[1] 'Exploring Teacher Mediation of Subject Learning with ICT: A Multimedia Approach' (2005-2007). Funded by the UK Economic and Social Research Council (RES000230825).

one-off continuing professional development (CPD) sessions may be evaluated at the time as enjoyable, there is little evidence that such learning is translated into the classroom (Muijs & Lindsay 2008). This appears to be because such a form of CPD overlooks the complex contextual differences between a course and the reality of teachers' own classrooms, particularly with regard to the type of software and hardware available, the type of technical support needed and the specific pedagogical issues associated with their own pupils' needs and their own subject demands. It is also attributable to lack of time and space for teachers to trial and embed new practices during busy working lives.

Instead, ongoing on-site learning within existing communities of practice (Wenger 1998) appears to offer a more fruitful form of professional development, in terms of changing practice. Such a 'situated perspective' (Putnam & Borko 2000) appears to offer opportunities for teacher learning to be meaningful by engaging with teachers' beliefs that subsequently lead to changes in practice (Lerman 2001). Initiatives based in such a perspective can also be designed to offer teachers a 'generative' rather than 'responsive' role (Jaworski 2001) in their learning, which may again contribute to successful change in practice.

Being involved in regular critical reflection on concrete examples of practice and sharing among colleagues can promote potential long-term changes in teacher behaviour (e.g. Scherer & Steinbring 2006). One way of encouraging this is to use 'peer learning', which refers to the way that teachers come together in a formal manner to mutually support each other's learning (e.g. Zwart et al. 2007). Our teacher development initiative used peer learning in conjunction with the stimulus of a multimedia resource to encourage teachers to share their ideas about teaching, as well as to experiment with new practice and observe the effects on students.

Overview of the resource and toolkit for our professional development initiative

The toolkit represents an adaptable model for professional development using a research-based multimedia resource. The T-MEDIA CD-ROM contains video clips showing one mathematics teacher's use of a data projector, laptops and graphing software in her classroom, with analytic commentary and built-in prompts for discussion. The resource encourages teachers to experiment and reflect on their own use of such technology in promoting students' mathematical learning, by providing stimulus material for collective debate. Teachers are thereby able to draw on – and critically evaluate – classroom practice from outside their schools as well as on ideas from colleagues, and to synthesise these with their own situated knowledge. The adaptable model surrounding the use of this resource therefore differs from conventional professional development activities in providing opportunities for reassessment and development of pedagogy that are sustained over time by teachers themselves – with support from subject and school leaders being critical.

A toolkit document, which includes a flowchart of the adaptable model of collaborative working together with suggestions for professional development activities and a senior management team briefing sheet, was designed to act as a guide for teachers working together with departmental colleagues through this reflective process, independently of outside intervention (Hennessy et al. 2008). It is based on an iterative cycle of teacher-led discussion and review of the stimulus material, lesson planning, peer lesson observation, collective reflection and refinement. Commissioned by the National Centre for Excellence in Teaching Mathematics (NCETM), the toolkit document is the outcome of trials in guiding use of the T-

MEDIA CD-ROM by an Advanced Skills Teacher (AST) with other teachers. Both the toolkit and the multimedia resource are hosted on the NCETM portal (www.ncetm.org.uk) and are thus freely available.

Design principles of the initiative

This professional development initiative was intentionally developed to be flexible, teacher-led, collaborative, and based on supported professional dialogue and reflection on practice that are ongoing over time. While the resource and toolkit have a clear focus on developing subject pedagogy, teachers choose whether or not to focus on the use of ICT; the range of foci they selected to explore during our trials clearly demonstrated that the issues raised also went well beyond the specific teaching context illustrated on the CD-ROM. In addition, the initiative proved useable with departments and teachers at all levels of (teaching/ICT) experience – in groups or individually – and, while providing a department with common purpose, it also served a wide range of different needs simultaneously. It offered new opportunities to lead, to carefully observe colleagues' practice and understand their reasoning, to take stock and to experiment with new pedagogical techniques, and to explore the potential of new technological tools. The process likewise offered a rare opportunity for teachers who are not performance managers to observe – and collaboratively deconstruct – a colleague's lesson. Teachers could benefit from planning lessons together, supporting each other in developing and trialling new ideas.

In sum, our initiative valued teachers' own aims, insights and motivation to improve pupil learning outcomes. It was based on the critique of real examples of practice in an ordinary classroom with far from perfectly behaved children.

Research design

This work examines the impact of pilot work and the subsequent trials of the 'toolkit' document on participating teacher groups and their wider subject departments, and considers this in light of the radical variation in levels of participant experience of teaching and technology use. We were interested in how the teachers considered that engagement with the project affected their practice in the classroom, their thinking about teaching, their working with department colleagues and their perceptions of pupils' responses.

Our research questions were:

- To what extent and how has engagement in this process influenced participants' classroom practice, either in relation to technology use or not? Were any changes reported in the practice of individuals, participating groups, or wider departmental communities?
- To what extent and how did involvement in the process stimulate any impact on pedagogical thinking?
- Was any change embedded in ongoing professional development practices, either at the departmental or institutional level?

The AST worked with a small group of teachers within each of two school departments (one his own), initially to encourage teachers to engage with the resource and then also to support observation in classrooms. The toolkit was developed as an outcome of these studies and then trialled by a third, small secondary department, in the presence of an academic researcher (Bowker) who observed its use.

Primary data collected included a series of meeting recordings, lesson observation notes and short follow-up interviews with some of the teachers at the time. Further data was collected a few months later and included teachers' reports of student feedback, responses to new approaches, and reflections on the contextual factors that affected how they were able to develop practice further.

The impact of the professional development activity

Teachers had a range of reactions to the resource, yet all of these were able to lead to change in practice. One teacher described the impact for him:

> I found it frustrating, the fact that she wasn't using the technology to its potential, I didn't feel. That's what's given me the impetus of thinking, 'well, OK, it's all right sitting criticising, but what would you do differently?' So I think I had a lot more interactivity in my lessons. [The pupils] were coming up and answering the questions on the IWB. And I'll be perfectly honest, I was actually quite surprised how well it went.

Our observations were that the typical initial reaction of teachers to watching the clips in the resource was to launch into a discussion of the teaching observed. Initially critical of what they were seeing – for example one teacher stated: "There is absolutely nothing that she did there that was particularly special" – these discussions developed as teachers reflected more deeply and shared their thoughts. Ultimately, teachers would comment on aspects they appreciated, for example "You've got to get the culture of working [among pupils] first, which is what Sarah had." And: "You can see it's obvious she knows what she's doing, so it's obvious it was a very good maths lesson." The focus of teachers while watching clips varied too. One stated:

> The technology wasn't the thing that struck me, really, it was the style of teaching and the interactivity with the kids which I don't do a lot of … and it was a real eye-opener.

Another remarked:

> I really liked her questioning and accepting a wrong answer… Is that the funnelling, when she was giving them alternative questions that kind of like clued them into the right answer?

Yet others were primarily interested in how technology might be used.

Comments about the different aspects of the initiative suggested that teachers valued the experience highly. Reflecting at the end of the project, one teacher remarked: "The [multimedia] resource is the trigger – it triggered the conversation and the conversation triggered the lesson." The peer reflective discussions were mentioned by almost all teachers, who considered them beneficial: "The sharing of the ideas in the group were what produced the lessons." One teacher pointed out that change in practice would not be instantaneous, but felt the initiative provided an impetus:

> I've tried doing a few things even in the last couple of weeks that without this stimulus I wouldn't have changed, but I'm very conscious that it's going to take a little while for me to hone my skills because it's changing a style of delivery.

Another, reflecting this need to continue to work collaboratively at changing practice, enthusiastically stated: "We must keep on doing this. It is essential we keep on doing this."

Returning after 6 months to the schools to conduct follow-up interviews, we were interested in any longer term benefits of being involved in this type of professional development initiative. All the teachers in these interviews alluded to using more technology in their lessons. One head of department stated:

> The biggest thing that's happened […] as a direct result of the project […] is that we've [got] 35 laptops which [...] all of us have used with most of our classes.

Another teacher, in a different school, announced that he had just been appointed to a new job on the strength of his new skills for using ICT in the classroom, directly derived from the meetings and the toolkit development. He was looking forward to implementing in his new institution more of the ideas with which he was still buzzing. He was 'incredibly grateful' to have been involved, not least because it was the only professional development he undertook during the 18 months posting at his current school.

Several teachers wanted to return to the resource and the toolkit: "I want to spend some more time with that initial resource"; "We sort of saw ourselves teaching. […] I'd like to do that more and that will also give me an incentive to design, hopefully, better lessons because I know that I'm recording myself." One teacher also envisaged how he would like to extend his experience to others who had not been involved: " 'Ok, this is what we did […] and then we start the conversation from there. It's like, 'well OK: wider implications of technology in maths: Let's go!'" While teachers had not formally returned to the resource and toolkit, owing to time constraints, one department had regularly continued the informal dialogue, swapping ideas about what might work with different year groups, and on occasion had been able to observe each other's use of technology. The findings will be further explored in a forthcoming longer version of this paper.

Conclusions

With support at the school level and from the toolkit, exploring a single but rich and flexible resource appeared to give the diverse group of teachers involved in the trials a means of opening windows on practice and moving both classroom and departmental practice forward in ways that they wanted to sustain over the long term.

While acknowledging the complexity of attributing motivating factors to 'the project', nonetheless teachers in each of these three schools reported tangible effects on their classroom teaching. For some, the project acted as an impetus for acquiring appropriate hardware or exploring software. For others it was an opportunity to experiment with different pedagogic techniques, to obtain feedback, or to discuss arising issues in a collegial manner.

These claims need to be contextualised since in all cases additional supportive factors could be identified. For example, impending curriculum changes encouraged the need to use technology, and pupil enthusiasm for using computers appeared to contribute to teacher motivation. These helped to maintain a forward momentum when faced with difficulties such as a lack of appropriate support in terms of allocated time to meet with colleagues.

In each school the process of reviewing and debating the video material together with colleagues, exploring some routes through the material suggested by the toolkit, trialling new approaches, and critically reflecting upon the outcomes, triggered demonstrable *changes in pedagogical thinking and practice – whether teachers were using technology or not.* Professional dialogue between colleagues working within an established and supportive community-of-practice proved central, and the desire to create further opportunities to perpetuate this dialogue was itself a key outcome for participating departments.

The findings indicated that the success of the collaborative professional development initiative needs to be interpreted, however, not only in terms of its generic underlying principles but also in light of a complex interplay of additional situated factors. These include teachers' personal and professional concerns about their own practice, responses to the diverse practices depicted on the T-MEDIA resource and observed in colleagues' teaching, classroom dynamics and needs of particular pupil groups. Our work thus directly supports Mason's premise that teachers "have to work on themselves, informed by research, and shared practices" (Mason 1994, 179). In sum the cyclical process of inquiry was effectively scaffolded by the initial guiding framework we provided, but rapidly took on its own momentum, resulting in a unique course of action in each case; participants spontaneously and enthusiastically developed their own fruitful pathways of learning.

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An Early Years' CPD initiative for mathematics: the power of collaborative, 'grassroots' learning

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Local 'grassroots' *Children's Mathematics Network* groups are initiated and 'owned' by teachers and practitioners and they explore and develop their understanding of *children's mathematical graphics* (Carruthers & Worthington, 2005; 2006; DCSF, 2008) in their own ways. New research findings reveal the effectiveness of this form of 'continuing professional development' (CPD) and its impact on children's mathematical thinking (NCETM, 2009). This paper explores the philosophy underpinning these groups, and their inter-connectedness with children's mathematical graphics.

Keywords: collaborative CPD; co-constructing meanings; children's mathematical graphics; impact; learning

Introduction

This paper explores the findings of our recent CPD initiative and considers the impact that the practitioners' involvement has had on their developing pedagogy and on their children's mathematics. It explores the philosophy that underpins and connects both *children's mathematical graphics* and our CPD initiative and draws some conclusions about practitioner-networks.

Through our research into young children's early 'written' mathematics we originated the term *children's mathematical graphics:* this describes the wide range of graphical marks and representations children use to support their mathematical thinking (Carruthers & Worthington, 2005; 2006). These graphics have their beginnings in children's imaginative play, (Worthington, 2009). In 2008 the *Williams Maths Review* (DCSF, 4) highlighted the importance of *children's mathematical graphics* arguing 'The review also plays great score by play-based learning of a mathematical nature, and makes specific recommendations regarding early mark-making as a precursor to abstract mathematical symbolism'.

We recognise the importance of supporting children's learning to uncover their own ways of thinking: this includes a respect for young children's ability to think mathematically; to initiate and play with ideas; to make decisions and take risks and solve mathematical problems that have personal meaning for them. We recognise that (as with mental methods) there are many ways of working rather than one 'right' way generating and rather than pre-determined written outcomes, encourage children to use pen and paper to explore their ideas, rather than for pre-determined outcomes. Children have 'ownership' of their mathematics and the emphasis is on making meanings through their mathematical representations and meanings are negotiated and co-constructed through collaborative dialogue. This led to a pedagogical shift in which teachers focus on the child's line of enquiry, adults listening and observing sensitively in order to understand the complexity of their emerging mathematical thinking. This perspective is rooted in socio-cultural and social-semiotic theories (Vygotsky, 1978; Kress, 1997) and results in an open, democratic approach to learning. We argue that *children's mathematical graphics*, and our local CM Network groups, inter-connect through their underpinning philosophies.

Mathematics CPD: reserch literature

It could be argued that one of the key determiners of an effective CPD model is that teachers make a significant 'concept shift' that impacts on their practice and has a demonstrable and positive impact on children's mathematics. In order to experience a conceptual shift, teachers must weave in and out of practice and theory: this requires time and shared socialization to supports common goals. There are likely to be a number of reasons for the limited impact of traditional models of CPD, including a lack of support for practitioners once they return to their setting and for continuing opportunities for reflection and enquiry. Whilst such conceptual shifts are what leaders of traditional models of CPD and national training for mathematics hope for, it appears that such outcomes are not always borne out by research.

Within schools and local authorities the majority of mathematics CPD appears to be a traditional 'delivery model', often termed 'training' and described by Drummond as 'learning by swallowing' (2007). Cooper & Boyd (1997) highlight the low levels of impact on teachers of this model and cite research by Joyce & Showers, (1996) and Glickman, (1993), who demonstrated that two months following a workshop, only 16-20 per cent of teachers made recommended changes. MacNaughton argues that for many early years educators, professional learning starves them 'of the nutrients that support them to proactively, enthusiastically and knowingly draw on leading edge theories to push the possibility for democracy and progressive social change in their lives with young children' (2005,190).

Involvement in teacher research has also been identified as impacting on classroom practice (Stenhouse, 1979; Slavin, 2008; Issitt & Kyriacou 2009;) and can be a valuable aspect of CPD. Yet it appears to be that many practitioners and heads of centres and schools do not necessarily recognise theory and research as rich resources of knowledge which can be used as a guide to practice. As Rodd (1994) suggests, practice is not sufficient in itself to constitute the whole curriculum design; research is equally important. The polarisation of practice and theory as two very separate entities is misleading: they depend on each other.

An alternative model for CPD

A review of 17 research studies into *collaborative* CPD found it 'was linked to a positive impact on teachers' classroom practice' (Rundell & Seddon, 2003, 3). Teachers 'shared a stronger belief in self-efficacy and reported a high level of commitment to change. Their enthusiasm for collaborative working and professional learning had increased and the recognition that peer support was beneficial featured strongly in many of the studies'. Johnson & Johnson propose 'The superiority of co-operative over competitive and individualistic learning increases as the task is more conceptual, requires more problem-solving, necessitates more higher-level reasoning and critical thinking, needs more creative answerers, seeks long-term retention, and requires more application of what is learned' (Askew & Carnell,1989, 43).

We also questioned the extent to which our provision of one-day courses had lasting impact on teachers' thinking and practice. Moreover, we believed that the philosophies and values we espoused in our work with children should underpin our CPD. In 2003 we established the *Children's Mathematics Network* (CMN), and subsequently introduced the concept of local *'CM Network groups'*. The focus of the Network is on '…*children's mathematical graphics* and the meanings children make…. Our aim is to hear the voice of the child and to support effective pedagogy for this significant aspect of mathematics in this phase', (*from CMN website*).

Dialogue

Dialogue is a significant factor in collaborative groups, (e.g. Mercer, 2000; Mercer & Littlejohn, 2007). Keiny proposes that the social contexts promote interaction within the group, leading to 'the exchange of ideas that stem from the teachers' practice (and) leads to the decontextualization of personal experience, and construction of knowledge of a more abstract nature. Reflection of ideas within the reflective group… turns strategies into meaningful pedagogic knowledge' (1994, 165).

Teaching young children is complex and practitioners need time to think, to allow their ideas to *meander* (Carruthers, 2008). During a study of the role of research in Children's Centres, the researchers realised that in their collaborative dialogue, they were 'meandering', defining this as 'reflectivity on common concerns where time is not a barrier'.

> We were finding pathways, in our own way, in our own time. We travelled down pathways and then started from the beginning to reflect our question further. It is cathartic and gentle and can help us see clearly because we do not feel pressure to have an outcome.' It is not 'outcomes' based: *'What if we do not go anywhere - and does it really matter because just once in our meanderings we could strike upon something significant*, (Carruthers, Journal entry, March 12th 2008, 16).

Carruthers proposes that teachers need to have opportunities to self-reflect and generate their own theories rather than be 'Passive victims of the education system – it is important that teacher's know beyond the government dictates' (2008, 6). In the following section we focus on three 'grassroots, early years 'communities of practice' for Early Years teachers and practitioners which foreground dialogue, socialization and co-construction.

Collaborative Early Years CPD

Example 1: The first example is the 'Emergent Mathematics Teachers' group of which we were founder members. This was a 'grassroots' group 'owned' and shaped by its members. Meetings were based at each others schools and later at our homes, and the social aspect of our involvement was significant in sustaining interest. The group's success led to its sustained development over 6 years and significantly, to increased feelings of personal and professional empowerment resulting in to self-generated and evolving theories. Our knowledge and developing pedagogies were supported by our research and underpinned by theories about mathematics which we read and through dialogue with leaders in the field of mathematics education. Eventually the two of us went on to research a significantly new aspect in depth, *children's mathematical graphics*.

Example 2: Support for an alternative view of CPD comes from a post-structuralist perspective which emphasises issues of power, knowledge and truth about how these issues relate to teachers and practitoners experiences in their work in early chidlhood settings in South Australia who established a 'critically knowing community'. They felt that 'important aspects of early childhood education were under attack and that the opportunities to argue differently about curriculum possibilities were limited or non-existent' (Barnes, in MacNaughton, 2005, 206). Barnes also acknowledges that they have demonstrated that educators 'do not have to wait for others to produce professional learning opportunities for them'; (in MacNaughton, 2005, 209).

Example 3: In the 1990s we were founder members of a local group, the *Emergent Mathematics Teachers*. We met through our interest in children's mathematical learning and our desire to effect pedagogical improvements. The group was self-directed and relevant to our children at the time: we drew on our experiences and the children's learning was at the heart of our discussions, influencing curriculum decisions we made. Our involvement in this group subsequently influenced our decision to introduce CM Network groups and the first of these was started (in 2007) by a teacher in Bristol who attended two of our one-day courses, which served as a useful introduction to *children's mathematical graphics*. The group attracted teachers and practitioners from the birth to 8 year age range, including mainstream and private nurseries and schools. Focusing on a new aspect of education they had chosen, and which stepped outside the 'official' curriculum, was seen as a positive experience. In turn, their developing pedagogy has had considerable impact on the children's mathematics, for example, several of the members highlighted the extent to which they valued modeling (socio-cultural, indirect adult; peer modeling and direct adult modeling; (Carruthers and Worthington, 2006).

The effectiveness of such groups has recently been independently acknowledged by the *Researching Effective CPD in Mathematics Education* (RECME). The overarching aim of the research was to investigate the interrelated factors that contribute to 'effective' CPD for teachers of mathematics and the outcomes of the study will inform future CPD and impact on policy-making in education (NCETM, 2009).

The final report includes 6 case studies to highlight various positive aspects and the CM Network group is one of these. The researchers noted that this initiative focuses 'on careful consideration and analysis of children's mathematics and the ways in which professionals can support and encourage the children's mathematical thinking' (NCETM, 2009, 65). The report observes that 'The standard of the mathematical understanding, thinking and reasoning that the displays revealed was far higher than the specified curriculum objectives for children of this age' (NCETM, 2009, 64).

For one of the teachers in particular, involvement in the group led to her 'shifting quite considerably from her previous practice and overcoming an initial reluctance to change and sceptisism about whether the change would be beneficial to the children's learning' (NCETM, 2009, 65). This is direct evidence of a teacher's significant *conceptual shift*, from previously relying on worksheets for all children's written mathematics, to supporting children's mathematical thinking through their own ways of representing their thinking. The report argues 'that ways of working with teachers that facilitate their mutual support and offer them ownership of the content, purpose and direction of their CPD may be particularly effective in supporting changes in professional practice that are radical' (NCETM, 2009, 65).

Equally significant was that 'The teachers reported how the research aspect of their CPD affirmed their perceptions of their teacher-self, leading to confidence in their professional self. They also reported how working on their existing interests and understanding led to a deepening development of their teacher-self and felt satisfying.' Furthermore it led to feelings of passion for their mathematics teaching: '*It has made me research an area of the curriculum about which I am strangely passionate, reflect on my own understanding and practice, collect and collate evidence and share this with fellow maths enthusiast within my school and the group'* (NCETM, 2009, 99).

Summing up the case study of the CM Network group, the report concludes:

> Participant ownership of this initiative helps to sustain involvement and that the members support one another in sustaining this passion and enthusiasm. Overall, the initiative supported the participants in their professional change by giving them a space for the detailed and joint consideration of children's mathematical thinking. It supported them in following up research sources that would support their analysis of the children's mathematical graphics and enabled them to encourage children to take charge of their own mathematical activity. It also offered them a supportive and encouraging arena in which their professional concerns and difficulties could be discussed, (NCETM, 2009, 65).

Conclusion

We argue that the traditional 'delivery' model maybe good at providing an introduction of a particular aspect of mathematics for teachers, but is less effective for embedding concepts or for a sustained conceptual shift. The recent RECME research findings suggests that democratic, 'grassroots' groups appear to have important advantages over the more 'traditional' delivery models for CPD, both in respect of practitioners' professional development and their impact on children's learning. Significantly, (with the exception of our local CM Network groups) the CPD projects researched in the RECME Project all received funding. However, we believe that funding can sometimes exert pressure on teachers to take actions for specific outcomes: they are obliged to commit rather than really want to.

The success of *children's mathematical graphics* in supporting deepened understanding of the abstract, 'written' language of mathematics is dependent on teachers and practitioners having time and opportunities to think things through themselves, to reflect and to critically analyse: these skills appear to be best nurtured in collaborative groups that are 'owned' and led by practitioners themselves. The CPD described here has been acknowledged as successful in supporting teachers and practitioners in developing their understanding of this important aspect of mathematics in the Foundation Stage and Key Stage 1. However, our conclusion is that through many current opportunities for mathematics CPD, teachers and practitoners' potential may be largely unrealised: as we have shown in this paper, there are other possible ways.

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Assessing the digital mathematics curriculum

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In their literature review of e-assessment Ridgeway, McCusker and Pead note an "emerging gap between classroom practices and the assessment system" (2004, 17-18). This gap threatens to undermine the effective development of ICT in the teaching and learning of mathematics. An examination of currently available on-screen assessments indicates that stand-alone instructional programs, designed to teach a specific set of skills or topics, are relatively well supported by tests composed of constrained item-types which can be computer administered and marked. On the other hand, tool software such as dynamic geometry or computer algebra packages may be neglected in the classroom because their use does not form a focus construct within the current assessment system. In this paper some of the constraints on test development that have led to this situation are explored, and ways in which tool software usage might be incorporated into an effective mathematics assessment are considered.

BSRLM Keywords: assessment; digital; mathematics;

Assessment and change in the mathematics curriculum

In their literature review of e-assessment Ridgway, McCusker and Pead (2004) observe that

> there is a danger of an emerging gap between classroom practices and the assessment system… [In] mathematics and science, the use of graphics calculators, spreadsheets, computer algebra systems (CAS) and modelling software is commonplace (and universal in professional practice). Assessment systems that do not allow access to these tools are requiring students to work in unfamiliar and maladaptive ways. Non-ICT-based assessment can be a drag on curriculum reform, rather than a useful driver. (2004, 17-18)

Other writers have drawn attention to this 'gap between classroom practices and the assessment system' in England (Pimm and Johnston-Wilder 2005, Wright 2005). A similar argument has been put forward internationally, particularly in relation to the influence of assessment on teachers' readiness to accept and use computer algebra systems (Lokar and Lokar 2001; Meagher 2001). So there is a common view that unless assessment changes to support and encourage the use of computers in the classroom learners may not acquire the mathematical skills they will need in the future. There is also a general concern, however, that this may be difficult to achieve, as compromise and a change of focus are needed.

Approaches to the digital assessment of mathematics

Weist has developed a classification of mathematics teaching software which distinguishes between *instructional* software, which is 'designed to teach students skills and concepts', and *tool* software, such as dynamic geometry packages or computer algebra systems, which may be 'used as an aid towards another goal' (2001, 46-47). As Papert put it, in the first case 'the *computer is being used to program* the child', while in the second, '*the child programs the computer*' (Papert 1980, 5). These two types of software make different demands on teachers and learners. Many instructional programs may be used as they stand, sometimes without any direct input from the teacher. Tool software, on the other hand, generally requires all users to build up their experience of the programs before they can use them effectively, and this takes time and commitment.

Just as mathematics *teaching* software can be classified into two general categories, so also two broad types of *assessment* task may be identified – although there is a lot of overlap, with some activities showing some of the characteristics of both. Scalise and Gifford developed a "taxonomy" of computer-based item types, ranging between those with

> fully constrained responses…. which can be far too limiting to tap much of the potential of new information technologies, and fully constructed responses…., which can be a challenge for computers to meaningfully analyze even with today's sophisticated tools. (2006, Abstract)

Thus on the one hand there are items that have one, or at most a limited and definable range, of correct responses, so they can be instantly and automatically marked by the computer. Tasks composed entirely of constrained items of this type are somewhat akin to the simplest type of instructional software program in that they can be used by the learner with little input from the teacher and can provide immediate feedback to both. The most extreme example of this approach would be a test consisting entirely of simple four- or five-option multiple choice items – but the range of item types that a computer can be programmed to mark automatically is much wider than this (Clausen-May 2005). The question shown in Figure 1, for example, from the World Class Tests of mathematics, makes good use of the capacity of the computer to provide interactive graphics to allow learners to explore a situation and search for a solution to a problem, but is none-the-less computer markable. In effect, this is a multiple choice item – but one in which every intersection on the grid is a possible option, giving a total of 144 options.

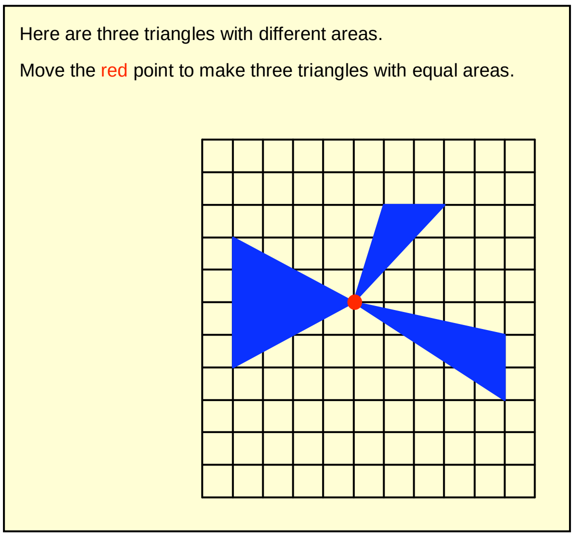

Figure 1 Threlfall and Pool (2004, 8)

Tasks that centre around the use of tool software, on the other hand, are likely to be more open, requiring constructed responses that leave many decisions to the individual learner. An assessment task of this type may involve significant elements of exploration, investigation and problem solving. The outcomes are therefore likely to be different for different learners, not just in terms of their success with different parts of the assessment but in the particular approaches and routes through the problem that they take. While it might be possible to program the computer to recognise and credit any completely correct solution however it was reached (as long as there are only a limited number of 'correct solutions'), it may be more difficult to award partial credit for a range of different responses which show some progress towards a solution but are not fully developed.

Constrained, computer-marked items on the one hand, and more open problems that may be explored using tool software on the other, offer different opportunities but also have different limitations. Teachers may welcome the support that a test composed of the former type of item can provide, offering immediate high quality data relating to the learner's knowledge and understanding of a range of mathematical skills and concepts. However, the restrictions imposed by the need for a closed set of possible responses may make this type of task less suitable for the assessment of such problem-solving skills as representing, analysing, interpreting and evaluating, and communicating and reflecting (Qualifications and Curriculum Authority 2007). These mathematical skills are highly valued, at least in the rhetoric of the school mathematics curriculum, but their assessment may demand a greater degree of flexibility than that offered by computer-marked test items.

Assessing mathematics with tool software

There is currently an explosion of computer-marked mathematics tests composed of items based in an *instructional* software mode. These range from simple multiple choice questions at one extreme to some of the very ambitious dynamic interactive items found in the World Class Tests at the other. The former can usually be created in a set format which makes them relatively cheap to develop, while the latter require individual programming for each item so they are likely to be significantly more expensive. Both, however, allow a closed set of possible responses, so they are computer-markable.

In contrast to these computer-marked tests, there is a dearth of well-constructed materials to support teachers in their assessment of learners' investigative and problem solving activities using commonly available *tool* software. This lack could relate both to pedagogical and to economic factors.

Threlfall and Pool observed how learners demonstrated aspects of ICT-specific learning in their approach to some of the dynamic interactive questions in the World Class Tests. The learners "played" with the mathematics in a way which differed significantly from their response to more closed, paper-based activities (2004, 15). So, the authors argue,

> success on some kinds of computer-based assessment items can arise from different skills and abilities than those required for success in related paper and pencil items. (2004, 11)

Threlfall and Pool's description of the exploratory approach taken by learners working on some questions in the World Class Tests of mathematics could offer a route to the assessment of mathematical reasoning and problem-solving (Threlfall and Pool, 2004; Threlfall et al, 2007). However, the skills that learners employ when they explore a piece of mathematics using an interactive program may not be accepted as valid constructs of a mathematics assessment.

> For the questions that do offer exploratory potential, there is… the issue of whether the qualities and skills that are used to answer them are felt to be legitimate criteria for the assessment being considered. (2004, 14)

So, for example, the use of trial and improvement in the solution of simultaneous equations is generally frowned upon in the classroom, and may lead to a loss of marks in a formal test or examination. However, as the authors observe,

> it is… possible that notions of legitimacy will change, and that the use of exploratory approaches and intuitive informal understandings will become accepted as desirable competences, and therefore valid criteria in assessment. In the …. context of World Class Tests, for example, they are seen as qualities that able mathematicians use to resolve challenging problems. (2004, 15)

Not everyone, however, will agree that this change in 'notions of legitimacy' is acceptable. In a discussion of a different selection of questions, taken from the World Class Tests of problem solving, Ridgway and McCusker (2003) argue that 'Too often, work is characterised by guessing, rather than by a systematic attack on the problem' (2003, 326). This rather critical comment on learners' "guessing" may present another perspective on the "exploratory approaches" reported by Threlfall and Pool. Thus a change in pedagogy, with test constructs that focus on reasoning and problem solving skills, may, or may not, be accepted by mathematics educators.

Furthermore, the development of any piece of tool software involves an enormous commitment of time and effort on the part of the developers – and the educational philosophy that drives this effort may not focus heavily on assessment. Even if it were possible to develop an assessment that could be administered, and possibly marked, within a tool software program, the authors of the software might not be willing to accept the pedagogical shift that this would imply.

On the other hand, economic factors may make test developers unwilling to invest in the development of an assessment that depended upon a software tool over which they would have little control. There would always be a danger that the software could become unavailable, or could evolve in a way that made the assessment unusable. Furthermore, there could be copyright issues if the assessment were to be sold commercially – and designing, programming and trialling a computer-based test is an expensive process, so costs would have to be recouped somehow.

There has thus been little impetus for either the creators of the programs or for established test developers to develop and trial tasks to assess learners' mathematical understanding using tool software. The creators may not see assessment as central to the achievement of their objectives, while test developers might not want to risk an undertaking in which they would have to rely on the continued availability of the software on which the assessment depended. But so long as there is no established way in which learners' mathematical achievement using tool software can be recognised and measured, these tools are likely to remain on the periphery of the mathematics curriculum. If the visions of the developers are to become a reality then there needs to be a credible way to assess learners' achievement with reliable, valid, manageable and markable tests.

One possible approach has been trialled in England to assess learners' ability to use tool software effectively in the context, not of mathematics, but of ICT. This involved the development of a 'walled garden' of programs created specifically for the assessment. This set of tools, which was developed for the English national ICT tests for fourteen-year-olds, includes a word processor, a spreadsheet, a presentation tool and a database. In each case the software was designed to be similar, although not identical, to the tools with which the learners were familiar in the classroom.

However, the provision of a similar set of generic software tools for mathematics could present problems. The range would probably have to include a dynamic geometry package, a spreadsheet, a data handling package, and a computer algebra system. Of these, only the spreadsheet has been developed for the ICT tests. Spreadsheets, however, are relatively homogenous, so learners who have had extensive experience with one are not likely to be seriously disadvantaged when they are required to become familiar with another. Tool software that is used to teach most other areas of mathematics, on the other hand, is commonly available in a range of guises. So, for example, *Cabri-Geometre* and *Geometer's Sketchpad* are significantly different (Mackrell 2004; Johnston-Wilder and Pimm 2005), so if a dynamic geometry tool were to be developed specifically for the assessment of mathematics then this would have to bridge the divide between these programs, and also give access to learners who had experience of other tools such as *GeoGebra*.

Furthermore, there could be a danger that teachers might 'teach to the test' by limiting learners' access in the classroom to the software that was available in the assessments. This would prevent learners from becoming familiar with the wider range of tools used in more advanced educational and in commercial and professional settings. This issue has already been raised in relation to the national ICT tests in England. For example, the BBC reported the complaint of one experienced head of a secondary school ICT department that the tests assessed

> "the old spreadsheet, word-processing and presentation applications which I was teaching 15 years ago, plus a bit of e-mail"…But to recoup the cost of developing the test the National Assessment Agency had said that it would continue "in its current form" until 2013. (BBC, 2007)

So a mathematics assessment based on a walled garden of software tools could be quite damaging in the long run. However, a commercial developer might see it as an attractive proposition, especially if, as with the ICT suite, the tools were associated with a high stakes national assessment that would guarantee their take-up by schools.

Moving forward

While there is currently a rapidly growing range of computer-marked, restricted-response tests, both economic and pedagogical factors have tended to discourage the development of tool-based assessments. However, there could be another approach to the assessment of tool-based mathematics. Assessments that start with the mathematics rather than with the programs might be developed, with learners selecting any available tool software for their completion. Each task would require a robust mark scheme to enable teachers to mark it reliably, providing results that would offer valuable insight into the learners' mathematical understanding using whatever tool software they were familiar with. These assessment tasks should not give rise to problems of copyright as no particular software would be specified.

For example, a simple item requiring learners to use any dynamic geometry tool that permits dragging to construct an irregular trapezium would assess their understanding of its defining properties – such as that two of its four sides must be parallel, but the other two need not be. The item would need to be informally trialled with learners using different tools to establish key points in their reasoning, but it seems probable that a generic mark scheme for this simple task might be developed, offering, say, two marks for a completely correct solution, or one mark for a construction that had some, but not all, of the relevant defining features. More significantly, perhaps, this result would provide a useful insight into the learner's understanding of the properties and construction of an irregular trapezium. This test item would not require any programming on the part of the test developer, but it would have to be marked by the teacher. However, marking would involve a simple check to see whether the construction lost any of its defining properties when its vertices were dragged. Similar questions, involving different constructions, could be designed.

There are a number of obstacles to the development of computer-marked assessments of tool-based mathematics – but written tests have always been marked by teachers. Perhaps our next challenge is to develop a set of mathematics tasks to be carried out using any appropriate tool software, but with marking guidelines that will allow for reasonably reliable, moderatable, teacher marking.

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Comparing Research into Mental Calculation Strategies in Mathematics Education and Psychology

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This paper argues for the importance of re-examining theoretical assumptions in research into mental calculation strategies and strategic thinking in mathematics education. By contrasting research into strategic thinking in mathematics education with that in cognitive and developmental psychology, three areas are identified where important details of the model of strategic thinking are left unexplored in education research while being dealt with more thoroughly in the psychological literature. The areas identified are: the positing of innate processes; the nature of memory; and the relation between conscious and unconscious mental processes. The status and reliability of introspective reports on mental processes are discussed as an illustration of the potential of research in psychology to further inform mathematics education research in this area.

Keywords: strategy, strategic thinking, mental arithmetic, National Numeracy Strategy

The National Numeracy Strategy (NNS), introduced in English primary schools in September 1999, gave increased prominence to informal methods or strategies and mental calculation. Doubts have been raised however about the success of attempts to teach mental calculation strategies or strategic thinking directly (Bibby, Askew and Hodgen 2003) as recommended in the NNS. In mathematics education there is a substantial body of research on strategy use and strategy selection in arithmetic in particular, although there is little consensus in the research community about strategic thinking as an aspect of mathematical thinking or about the value of the concepts 'strategic thinking' and 'strategy' for studying mental arithmetic (Threlfall 2002). Some views of arithmetic learning, although not necessarily inconsistent with strategic thinking research, focus more on number sense, mental models, or conceptual understanding. Additionally, the individualist view of mind and learning suggested by strategic thinking is challenged by socially centered views of learning such as social constructivism, social semiotics, situated cognition and hermeneutics.

The study

This research project (Craig 2008) sought to compare research into strategic thinking and mental calculation strategies in mathematics education (ME) and developmental and cognitive psychology (PSY) in terms of method and the assumptions about strategic thinking being employed. The study was motivated by a sense of ambiguity in the use of the terms strategy and strategic thinking in ME: much research into mental calculation strategies in ME seems to assume an unspecified model for strategic thinking whereas, in PSY, although research into strategic thinking shares some common assumptions, there are in fact many variations on the basic model and discussion of these differences is more common. I will provide a brief overview of strategic thinking research in ME and PSY. I identify some overlooked assumptions about the nature of strategic thinking in research in mental arithmetic in ME: these assumptions, and their potential relevance for ME, are examined through a brief description of their treatment in PSY.

Strategic thinking in mathematics education and psychology

The term strategic thinking (ST), loosely, refers to the mental processes involved in making choices about different possible courses of action. Two common elements of most definitions of a strategy are that it is one of several options, and that it is directed towards a goal. Additional elements which are sometimes included in the definition by researchers are that the decision to apply a particular strategy and/or the operation of the processes involved in executing the strategy be conscious or at least accessible to conscious awareness. The nature of the implied difference between a mental calculation strategy and a procedure for calculation is not always made clear and there is, in general, some ambiguity in the way the terms ‘strategy’ and ‘strategic thinking’ are used in the ME research literature.

Differing research goals in PSY and ME influence the aspects of ST considered and the research methods employed. Three related research goals link ME research and PSY research: understanding cognition or the mind; understanding the development of the mind; and understanding learning. Understanding the development of the mind is the area were the two disciplines’ concerns overlap most. In ME an important goal is to understand processes of teaching and learning, and understanding the development of the mind (particularly in relation to mathematical understanding, knowledge or skills) is seen by some, although not all, as an important part of that goal.

Key to the theory of ST in PSY is the observation that individuals adapt their behaviour strategically (i.e. in ways which increase their likelihood of successfully achieving goals) to regularities in their environment. Research focuses on developing and testing models of ST as a tool to understand and explain this observation, which is accepted as holding across a range of areas of behaviour. The research goal is to model strategy choice by specifying the factors which affect it and how choices are made. Laboratory-based experiments are used to explore group behaviour in problem-solving situations under various conditions. ST models (sometimes developed into computer models) aim to account for behaviour, such as the variation observed in strategy use, by replicating it, and thus to demonstrate a sufficient mechanism for increasingly adaptive strategy choice. Research in developing and testing models of ST in PSY tends to assume that the individual, and the mind of the individual, can be meaningfully studied and discussed in relative isolation from any particular situation or actions.

Research into strategy use in ME is less uniform in terms of goals than that in PSY, and consequently in the role of ST models in the research. In ME, ST has been described as “developing a repertoire of mental and written calculation strategies and informed decision making about the use of these” (Kyriacou and Goulding 2004). ST is studied as an aspect of mathematical competence – as something which it seems that successful students and adults do. Research interests are in how this competence develops, what factors affect the acquisition of the competence, and in particular how teaching can encourage or develop ST. This clearly overlaps with the concerns of PSY.

In ME, as in PSY, much research has been devoted to observing and categorizing the different types of strategy children and adults use on basic arithmetic calculations and on exploring the factors which may influence this (e.g. Baroody 1987; Torbeyns, Verschaffel and Ghesquière 2005; Selter 2001). There are some differences in the methods used, with ME research more likely to employ realistic situations and classroom observation than PSY. The difficulties experienced by some students in mastering mental calculation, and mathematical disabilities have been interpreted in terms of strategy selection models (e.g. Ostad 1998; Roberts, Taylor and Newton 2007). In some ME research adaptive strategy use has been characterized as a *goal* for direct teaching: different methods are used to encourage pupils to use a range of methods, to apply them appropriately to situations, and to respond adaptively to new situations (e.g., Selter 2001; Beishuizen 1993).

Models and assumptions about strategic thinking

The focus, in PSY, on understanding underlying mental processes leads to a ST model which problematises some aspects of ST simplified or taken for granted in ME. An examination of the theoretical literature in PSY revealed three issues which remain relatively unexplored in ST research in ME and which are relevant to the concerns of education. The treatment of these issues in PSY could provide a starting point for their discussion in ME. Although they are not the focus of this discussion, there are of course ways in which ST research in PSY would benefit from increased awareness of research in ME, for example, in terms of the way in which strategy choices and individual development play out in actual classrooms.

Memory

Models of ST in ME and PSY differ in the extent to which fact-recall and memory are problematised. In solving arithmetic calculations, particularly with smaller numbers, one option is to recall the answer from memory. In ME research the individual is implicitly characterised as having some number facts which they know and hence will state in answer to a question (although they may make mistakes) and others which they do not know. The decision to recall a number fact rather than recalculate it might be assumed to be unproblematic either on the grounds that there is no significant time delay or cognitive effort required in recalling number facts, or because individuals are assumed to have the self-knowledge to judge quickly whether or not they have a particular fact stored in memory. Rarely are these options discussed in ME accounts of ST, although measures of working memory capacity and knowledge of memorised number facts are considered important both in mental arithmetic research and in ME more generally (see, for example, the research reported in Dowker 2004).

In contrast to ME research, ST research in PSY treats mental recall of number facts as a strategy option of equivalent status to any other option, the merits of which must be evaluated and weighed up by the individual before it is settled on (see, for example, Siegler and Araya 2005). An interesting implication of this is that it assumes the ability to internally assess the accuracy of mental recall – usually considered as unconsciously held knowledge in PSY. For ME research an interesting question which would arise from a consideration of this would be whether incorrect judgments of the accuracy of memory contribute to poor strategy choices, and how these judgments are formed. Another interesting subtlety in the psychological model of ST which is lost in current research in ME is the importance of goal-related factors in even this seemingly simple choice to recall an answer from memory rather than to calculate it afresh. The role of goals, potentially including social and affective goals, in determining strategy use has the potential to provide a different interpretation of some mathematical difficulties: in some ST models particular goal-orientations can be seen to encourage short term mathematical behaviour which does not promote long-term arithmetic development. Of course, this is not a novel insight for ME generally, but here we see the potential to build it into a cognitive account of strategy choice and mental arithmetic.

Innate vs. learned

In ME, ST is sometimes characterised as a desired end point to learning or as a teaching goal (see, for example, Heirdsfield 2000). In contrast, some psychological models characterise ST as an innate (or at least early developing) mechanism which drives later development of improved decision making, and accounts for the observed variability of strategy use in learners (e.g. Siegler and Araya 2005). As an innate mechanism ST is understood to be an equally valid description of the mental processes of those who succeed and those who struggle with mental calculation. In this model, the individual does not learn to generate appropriate strategies, nor does he/she learn to balance the relative merits of those strategies with respect to goals, rather these processes are innate, and they are said to guarantee (with some caveats) the development of increased competency in mental calculation. ME research in ST generally assumes that we can teach individuals to make 'better' choices whereas that in PSY assumes that the individual is necessarily making the 'best' choices already, given their goals, knowledge and competencies at a particular point in time, and failure must be understood not as failure to make the best choices, but as failure in the pattern of choices, over time, to contribute to development. The implications for teaching of the two perspectives are very different.

Conscious control of mental processes

In ME, it is assumed within many studies that the decision process itself can be improved by increased awareness of the different parts and that each of these is susceptible, in principle, to change and improvement. For example, discussion is considered to be important in increasing conscious awareness of the relative strengths and weaknesses of different strategies and of the problem factors which are relevant to the choice of strategy; this assumption of openness to direct conscious control and improvement underpins the teaching objectives suggested in the NNS.

Although researchers in PSY take different positions on the amount of conscious awareness and control possible over different aspects of ST, their description of the choice mechanism does not imply conscious control. This is not to say that an individual cannot consciously call to mind possible courses of action, weigh the alternatives and choose between them – but this is not the process generally labelled ST in PSY research. There is an important distinction to draw for education research: the question needs to be asked of how conscious processes of ST, such as those sometimes assumed in ME, and the largely unconscious processes studied in PSY relate to one another.

It should be said that, like ME research, much research in PSY is not explicit about which elements of ST are to be considered innate and which learned, and consequently which might be susceptible to alteration or improvement, and which elements, if any, are considered to be open to conscious awareness or control. In fact many subtle distinctions are contained in the use of terms such as conscious, unconscious, innate, learned, and articulable which need to be explored. Differences in the way mental mechanisms, drives or structures relevant to ST are conceived in ME and PSY have important implications for the educational consequences of ST research. Currently a lack of clarity in the ME literature about the role of unconscious processes means that such issues often go unaddressed, although the importance of ideas such as tacit knowledge and pre-verbal processes in early understanding of number has been acknowledged in other areas of ME (e.g. Wynn 2000).

Introspective reports

The importance for ME of examining assumptions about the model of ST being employed, and the potential of PSY to provide a starting point in that exploration, can be seen in the use of introspective reports as data. Verbal or introspective reports from individuals on their own mental processes are the primary data source for much research into mental calculation in ME. Their validity and reliability rest on theoretical questions about the type and degree of access individuals have to their own mental processes in general and to the processes suggested in ST models in particular, and also on practical questions about how to access this knowledge. Although many researchers address the reliability of data gathered from verbal reports as a standard part of their research it is not usually explored as a theoretical issue which relates to the model of ST being assumed. In PSY, a great deal of debate and research has centred on the validity and reliability of verbal reports (see, for example, Nisbett and De Camp Wilson 1977; Gaillard et al. 2006). This debate could be a valuable resource for ME researchers in beginning to explore the implications of using introspective reports as a theoretical issue in the nature of ST.

Conclusion

Any study of the mental processes involved in strategic thinking makes assumptions about what strategic thinking is and more generally about the mind. Ontological assumptions are made about the type of objects which are appropriate to study (although the status given these objects may not be made explicit). Epistemological and methodological assumptions are made when relying on introspective reports of mental processes: assumptions about the type of introspective knowledge it is possible to have and the ways in which it can be accessed. It is important that these assumptions are acknowledged and discussed in the research in which they are made, both in order to reduce ambiguity, and because, as I have argued here, some of the assumptions have educational implications. The treatment of mental recall in cognitive and developmental psychology raises questions about the place in strategic thinking models of individuals' knowledge of their own memory processes. The distinction between innate mechanisms and learned behaviours demonstrates the need to reconsider the assumption that all elements of strategic thinking processes are open to alteration through teaching and experience. Finally the nature of and complex relationships between conscious and unconscious, attended-to and unattended-to mental processes is of real importance in understanding mental arithmetic competence from a cognitive perspective. Psychological research in strategic thinking provides a rich source of discussion and research in this area which could act as a stimulus and resource for similar discussion in mathematics education.

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Primary pupils in whole-class mathematical conversation

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Although plenary sessions are common to mathematics lessons, they are often characterized by traditional approaches that endorse the position of mathematics as a kind of received knowledge and the teacher as sole validator of students' input. A socio-constructivist view of mathematics calls for a more conversational style of interaction among participants. In this paper an account will be given of a lesson in which children aged 9 – 10 years calculated the sum of integers from one to one hundred. Particular attention will be paid to one pupil, Anne, and her reassessment of a conjecture that she made early in the lesson. I suggest that particular teacher 'moves' facilitated engagement of other students with her idea and that this was one factor that led to her new insight.

Mathematics as conversation

The traditional classroom interaction structure is the Initiation – Response – Follow-up (I-R-F) model in which the teacher initiates an exchange and the student then makes a contribution and the teacher then makes a follow-up move (Sinclair and Coulthard 1975). In situations where the follow-up move is 'evaluative', the pattern is describes as I-R-E (Mehan 1979). It is suggested that the I-R-E structure reinforces the asymmetry of power between teacher and pupils (Mercer and Dawes 2008; Pimm 1994) – the teacher retains the locus of control and the students do little more than infer what is in his/her mind. Mathematics because of its association with recall of procedures is particularly susceptible to the I-R-E structure. Based on her observations of mathematics lessons, Wood (1994) describes a funnel pattern of interactions as one that involves repeated cycles of I-R-E where students are provided with leading questions in an attempt to guide them to a predetermined solution procedure. It is similar to the 'elicitation' pattern described by Voigt (1995) in which students, although given the opportunity to offer different solutions, are guided by the teacher to one definite argument.

The recognition that mathematics needs to be 'co-constructed' by students and teacher has led to interest in how a participatory model of discourse might be developed. In particular, it is felt that a follow-up other than evaluation might lead to a more conversation-like genre than stems from the I-R-E model. Nystrand and Gamoran (1991) use the term 'uptake' to describe the process of incorporating student responses into subsequent questions. They make a distinction between test questions which are designed to assess if the student knows what someone else thinks or has reported and authentic questions which signal a teacher's interest in what the student thinks. When the teacher uses authentic questions s/he opens the floor to what students have to say and this leads to substantive engagement by students. Related to the use of 'authentic' questions in mathematics lessons is the 'focusing' pattern of interaction identified by Wood (1994). In exchanges of this type, the teacher draws students' attention to the critical aspects of the problem giving them the responsibility of resolving the situation. A different type of follow-up is that of 'revoicing'. It is described as "the reporting, repeating, expanding, or reformulating a student's contribution so as to articulate presupposed information, emphasize particular aspects of the explanation, disambiguate terminology, align students with positions in an argument or attribute motivational states to students" (Forman and Larreamendy-Joerns 1998, 106). A key part of the reformulation is the use of 'So you think that' or 'So Tom thinks that'. O'Connor and Michaels (1996) describe this as a layering or lamination of the teacher's phrasing onto the student's contribution. It is akin to what Rowland (2000) terms an 'attribution hedge' in which some degree or quality of knowledge is linked to a third party. This layering, according to O'Connor and Michaels, "animates the student as the originator of the intellectual content" (1996, 79) and "makes possible an expanded and more contrapuntal set of voices and participant roles in constructing an idea than does the IRE" (p.97). Coupled with the attribution 'you think', the discourse marker 'so' opens up a new slot in the conversational space, giving the student an opportunity to comment on the correctness of the revoiced utterance. The effect of this 'layering' is to bring students' ideas in contact with each other and thus to effect involvement of all children in the conversation (O'Connor and Michaels 1996; Rowland 2000).

In a study in which four secondary school mathematics teachers were observed and videotaped for two weeks, Brodie (2008) used 'follow-up' as a key category to describe a teacher move. She maintains that it is broader than 'uptake' as it can refer to a contribution made by a learner either immediately preceding or some time earlier in the discussion and also because it includes both teacher-directed and 'conversational' moves. She found several ways that the follow-up move could be used by teachers and using some of the research cited above developed the following subcategories:

- Insert: The teacher adds something in response to the learner's contribution. She can elaborate on it, correct it, answer a question, suggest something, make a link etc.
- Elicit: While following up on a contribution the teacher tries to get something from the learner. She elicits something else to work on the learner's idea. Elicit moves can sometimes narrow the contributions in the same way as funnelling.
- Press: The teacher pushes or probes the learner for more on their idea, to clarify, explain more clearly. The teacher does this by asking the learner to explain more, by asking why the learner thinks s/he is correct, or by asking a specific question that relates to the learner's idea and pushes for something more.
- Maintain: The teacher maintains the contribution in the public realm for further consideration. She can repeat the ideas or ask others for comment or merely indicate that the learner should continue talking. Revoicing fits in this category.
- Confirm: The teacher confirms that s/he has heard the learner correctly. There should be some evidence that the teacher is not sure what s/he has heard from the learner, otherwise it could be press.

Brodie sees these moves on a continuum of more to less teacher intervention; in 'insert' the teacher makes his/her own contribution while 'confirm' and 'maintain' serve to keep the learner's input in the public domain. For these reasons, 'insert' and 'elicit' are viewed as more traditional than the other subcategories. However the teachers in Brodie's study, although committed to 'inquiry' mathematics were found to use a mixture of follow-up moves in any one lesson. This is unsurprising as teacher 'telling' is sometimes necessary and desirable (Chazan and Ball 1999). In this paper, however, it will be shown how a mathematical conversation dominated by 'press' and 'maintain' (in particular, revoicing) moves by a teacher afforded space for young students to engage with each other's ideas.

Background

The aim of my research is to investigate the factors that contribute to the development of mathematical insight by primary school pupils. The methodology is that of 'teaching experiment' which was developed by Cobb (2000) in the context of the emergent perspective and in which students' mathematical development is analysed in the social context of the classroom. For a period of six months, I taught mathematics to a class of thirty-one pupils (seven girls and twenty-four boys) aged 9 - 10 years. The school is situated in Ireland in an area of middle socio-economic status. Many lessons took place over two or three consecutive days, each period lasting forty to fifty minutes. I visited the class on a total of twenty-seven occasions. All phases of the lesson were audiotaped. When children were working in pairs, audio tape recorders were distributed around the room.

Forman and Ansell (2001) contend that analysis based on isolation and coding of individual turns is too limited to bridge the individual and social. Therefore, I conducted ethnographic microanalysis, which according to Erickson (1992) is especially appropriate when the character of events unfolds moment by moment. The approach adopted was top-down starting with the molar units (lessons) and moving to progressively smaller fragments.

The lesson described here took place on a third consecutive visit to the class during a week of the Spring term. On the previous two days, the pupils had been working on a lesson entitled 'Chess', a version of the 'handshakes' problem. The object of the activity was to find the minimum number of games that could be played in a competition where each player had to compete with all other participants. At the conclusion of this lesson some pupils had found the number of games necessary in the case of one hundred participants (i.e., the sum of 1 - 99) by using a calculator while others had latched onto the discovery made by one pupil, David, that the solution could be found 'by multiplying by the number less than it and halving it' ((100 x 99) ÷2). It was my intention on the third day to begin a new lesson but first told the story of Gauss (the mathematician who, as a boy, had amazed his teacher by his rapid calculation of the sum of integers from 1 to 100) in order to see if the pupils would make any connections between it and the chess problem. I expected that talk on this problem would last no longer than five or ten minutes. However, a rich discussion followed in which I had to 'let go'. The format of the lesson was a whole-class introduction, small group work followed by whole-class discussion The focus of this paper is plenary that took place in the introductory phase. The following transcript conventions are used: T.D.: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings;…: a hesitation or short pause; […]: a pause longer than three seconds; ( ): inaudible speech; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

Enactment

In the plenary under discussion the following thematic units or phases were identified:

- Phase One: Summing to one hundred, e.g. fifty plus fifty or five twenties.
- Phase Two: Finding partial sum and multiplying by appropriate factor (e.g. adding one to ten and multiplying by ten).

- Phase Three: Reasoning that solution strategy suggested in phase two would yield an incorrect solution.
- Phase Four: Adding '100' pairs (e.g., 99 + 1; 98 + 2, etc.).
- Phase Five: Adding sums of decades (e.g., 91 + 92 + 93 + …+ 100 and making appropriate adjustment to find sum of other decades).
- Phase Six: Applying solutions found for 'Chess' problem to Gauss problem.

Anne made contributions to the discussion in phases two, three, four and six. In phase three she underwent a change of mind about a solution strategy that she had proposed in phase two and this seemed to be on the basis of contributions of other pupils in the class. As pupils made suggestions, I wrote them on the blackboard. The following discussion took place during phase two and concerns an input that Anne made after Barry had suggested that the solution could be obtained by multiplying fifty-five (the sum of one to ten) by nine. Teacher moves are coded using Brodie's categories (above).

	Teacher moves
50 Anne: Thirty multiplied by ten.	
51 T.D.: Thirty multiplied by ten, why would you say it's thirty? Now Barry is using fifty, and he was multiplying fifty by about ten or nine, is that it?	Revoice (Anne) Press Revoice (Barry)
52 Barry: Yes.	
53 T.D.: And you think thirty multiplied by, why do you think thirty?	Revoice (Anne) plus press
54 Anne: Because if you add from one up to ten it's thirty.	
55 T.D.: How do you know if you add one up to ten it's thirty?	Press
56 Anne: If you add one to five, that's fifteen…	
57 T.D.: Hm, hm	Confirm
58 Anne: and then fifteen and fifteen is thirty so then if you multiply that by ten.	
59 T.D.: Ok, possibly that would get it for you. Fiona?	Maintain

Prior to Anne's suggestion, Barry had proposed that the answer would be around four hundred and fifty or five-hundred by multiplying fifty-five by nine. It is most likely that he was using the answer obtained on the previous day for sum of one to ten but I had erroneously thought that he was using fifty as a 'half-way point'. In turns 51 and 53 above, I revoiced the conjectures of both Anne and Barry. I also pressed Anne for justification. These moves probably assisted Anne and other class members to see the status of her contribution (and Barry's) as a conjecture – provisional, tentative and modifiable (Rowland 2000). Once Anne had justified her solution it was left in the public domain as one *other* possibility (see turn 59). Fiona then conjectured that the solution could be found by summing to fifty and doubling or summing to twenty-five and quadrupling. Thereupon, Alan commented on Anne's idea as follows:

	Teacher moves
66 Alan: Em, well, I don't think Anne's one is right.	
67 T.D.: Why?	Press
68 Alan: Cos ninety plus ninety eight plus ninety seven plus ninety six to ninety would be around over five hundred and when…	
69 Ch: Oh	
70 T.D.: Ok, so you are thinking that, you think, you disagree with Anne because you are thinking, what Alan is doing now, Alan is thinking ninety –[ ]- you are thinking ninety plus ninety one plus ninety two plus ninety three would give you approximately how much?	Revoice (Alan) Rebroadcast
71 Alan: Em, I don't know.	

72 T.D.: But it's… 73 Alan: But it would probably be over five hundred. 74 T.D.: It would be over five hundred, so in that section, if you are thinking about all those numbers there that would give you about, even just adding ninety to a hundred so you are thinking that would give you about five hundred- [ ] Barry? 75 Barry: Eh, well, I disagree with Anne as well because eh I counted, I counted up all the numbers up to ten and I got fifty five.	Press Revoice

Alan has observed an error in Anne's reasoning on the basis that the sum of numbers between ninety and one hundred would be 'over five hundred'. My revoicing in turn 70 is directed initially at Alan ('so you are thinking') to ensure that I understand him correctly and then to the rest of the class ('Alan is thinking') for the purpose of rebroadcasting his contribution. In turn 75, Barry indicates that he also disagrees with Anne – he makes reference to his earlier thinking that the sum of numbers between one and ten is fifty-five. The conversation than turned to consideration of the sum of numbers between ninety and one hundred after which Anne interjected:

	Teacher moves
91 Anne: I don't think … my answer wouldn't work. 92 T.D.: What were you thinking your answer was? 93 Anne: I thought it would be thirty multiplied by a hundred. 94 T.D.: Why would it not work? 95 Anne: Em, because you would have to, cos I did eh one plus two plus three plus four plus five and then em I got fifteen and then I added fifteen and fifteen equals thirty but then it would be more because you would have to add six, seven and that ()	 Maintain Press

In turn 91, Anne is reassessing her earlier reasoning and appears to have reached a new insight. Her reasoning is based on the fact that proportional reasoning cannot used to find the sum of consecutive numbers (in this case the sum of numbers between one and ten). While my question in turn 92, could be viewed as 'press' (for recall of a procedure), it serves to rebroadcast Anne's conjecture and thus has been categorised as 'maintain'. Although Anne's line of reasoning is different to that of either Alan or Barry, it is likely that their input caused some perturbation in her thinking.

Conclusions

Most of the teacher moves in this lesson were either 'press' or 'maintain' (usually revoicing). These moves served in this instance to make ideas public so that pupils became evaluators of each other's input. There was a sense that the pupils felt free to comment on the ideas of their peers but these comments were not viewed as disrespectful by the contributor. This is evidenced by the fact that Anne seemed to have little difficulty taking part in the conversation after Alan and Barry had discussed her input. However, in another situation, a different outcome may have emerged and thus the decision to 'go with the pupils' rests on the teacher's judgement. In this regard, Alrø and Skovsmose (2002) suggest that risk can be negative as one's suggestions may be refuted but it also includes the possible excitement experienced if one's input plays a significant role in the solution process. They advise that, in an educational setting, it is important that a balance is created between the negative and positive elements of risk. In this case it seems that there was a viable balance between the two and the resulting conversation allowed for new mathematical insights to be constructed – not only by Anne but by other pupils in different parts of the lesson.

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Socio-constructivist and Socio-cultural Lenses on Collaborative Peer Talk in a Secondary Mathematics Classroom.

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This paper uses socio-constructivist and socio-cultural lenses to examine transcripts of pupils' peer talk recorded while they were undertaking open-ended mathematical tasks in a naturalistic classroom setting. I discuss the two theoretical frames and then present episodes of peer talk from pupils between 12 and 14 years old which demonstrate how a socio-constructivist view of the zone of proximal development is enacted, and how a socio-cultural lens offers a window on social aspects of these established working groups which serve to provide the necessary support to enable all members of the group to access the mathematical knowledge being constructed.

Keywords: socio-constructivist, socio-cultural, collaborative, group work, socio-mathematical, norms, zone of proximal development

The socio-constructivist theoretical background

Vygotsky (1978) argued that the relationship between language and thought was a direct link, and that cognitive development was a social, communicative process. He interpreted individual utterances as having a role in both thought and language. Thus, a word implied both a generalisation of thought and a social interaction. He described the social construction of knowledge within a zone of proximal development (ZPD). In a classroom situation, the *actual* developmental level can be determined by traditional question-response-evaluation sequences and therefore *described*. The *potential* development, however, can only be *explained* rather than described because it is a process observed in relation to working with others.

Kinginger (2002) supports the use of Vygotsky's zone of proximal development in educational situations. She argues that Vygotsky's model of a process of cognition emergence has given direction to a more 'prospective' (rather than 'retrospective') educational emphasis, whereas more conservative discourses claim the same model as a "locus of transmission and reproduction of educational practices" (p241). Her argument for the advocacy of the ZPD is that it encourages collaboration and 'co-authoring' in learning. Such a model suits the research undertaken in this study, as in Kinginger's terms, this ZPD goes a long way towards supporting the 'prospective' educationist's implementation of social aspects of the construction of individual experience. Kinginger's interpretation of Vygotsky's model is of a "dialectic unity of learning-and-development" which includes the framework, the learning setting and the necessary resources, including those that are "dialogically constructed together". The outcome of viewing Vygotsky's model of learning as a dialogical process is that it allows a dynamic assessment of educational potential which includes mediating sources for its development rather than the more conservative static models of assessing potential. Learning in this context is seen as more than transference of new knowledge from inter-individual to intra-individual. It is seen also as societal change, in which "new forms of social activity are generated through joint cooperative action". The ZPD is seen as "an interactive space that holds potential for multiple – and unpredictable – transformations of human identity, of the culture's toolkit, and/or of the activity setting" (p246).

While Vygotsky's model may be interpreted as relativist, Wegerif (1998) argues that Vygotsky, like Piaget, believed in "a single rationality and a single progressive path of development" (p83). This, Wegerif claims, is because the basis of Vygotsky's view of knowledge development came from a Marxist interpretation of the world – that individuals are products of their social and historical influences. Thus, a Vygotskian interpretation of the educational effects of learning in the ZPD is not related to post-modern notions of enculturation but as an upward movement on a predetermined ladder of knowledge. His understanding of mathematical concepts was that they "represent essential aspects of an objective world" (p86). Wegerif claims that both Piaget and Vygotsky shared monological views of reasoning in which the principle of identity is central. He challenges this notion of a monological view of reasoning arguing that there is evidence in dialogue that identity of every sort is constructed – that reasoning is dialogical. Consequently, it is dynamic and affected by its environment. Dialogical reasoning is not established through identity but instead through differences, particularly those between "participants in dialogue" (p79). It is generated through conflict and takes the form of constructive argumentation between discourse participants.

The socio-cultural theoretical background

Much of the theoretical basis for a pedagogic approach using small group work in classrooms comes from the socio-cultural field. Collaborative group work (and research in this field), in which pupils work jointly on the same problem, is linked with ideas such as situated cognition, scaffolding, and the ZPD. As Coles (1995, p165) describes, "The social interactions developed in this kind of enquiry stimulate members of the group to think together; from a psychological point of view this pushes forward the level of thinking of each child and 'scaffolds' his or her cognitive processes". Although a Vygotskian view of learning encompasses a broad spectrum of contexts, it focuses on the individual outcome via an interpersonal process. Classroom studies with a socio-cultural framework (for example, Mercer and Fisher 1997, Wegerif 1998) have shifted this focus to an understanding of the process of learning within groups of individuals in specific social contexts. The focus here is on the interpersonal relations and their effect on intrapersonal learning within a group objective. These new units of analysis support a means of interacting which involves the whole self and a view of the interactions of a group as a means of cognitive development. Mercer (1995) proposes three necessities for this socio-cultural theory:

> A theory of the guided construction of knowledge in schools and other educational settings must do three closely related things. It must:
> - explain how language is used to create joint knowledge and understanding;
> - explain how people help other people to learn;
> - take account of the special nature and purpose of formal education. (p66)

This theory of the 'guided construction of knowledge' depends on two essential features – talk as social action, and the relationship between context and continuity. He contends that knowledge exists as a social entity, not just as an individual possession and that the essence of human knowledge is that it *is* shared. This gives recognition to how people construct knowledge together. "Individually and collectively we use language to transform experience into knowledge and understanding. It provides us with both an individual and a social mode of thinking" (p67). This model of talk involves learners in working towards a joint understanding through argument as an active process, rather than a mere pooling of information.

Mercer asserts that if a theory such as the one he offers for the 'guided construction of knowledge' is to explain how talk is used to create knowledge and understanding, it must incorporate context and continuity. In this sense, context is taken to mean the broadest interpretation possible for context – beyond the physical setting into the interactions between participants which develop the context. His interpretation of continuity also goes beyond a linear continuous path to mean the fluidity of change and a dynamic interactive flow. "If context and continuity are not well established in a conversation, the thread of a developing joint understanding may be broken and misunderstandings are likely to arise" (p68).

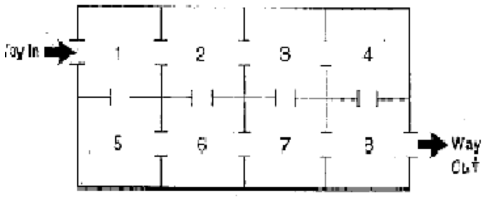
The study setting

The study was undertaken in an inner city secondary school (11-16 year olds) in the south of England in which 22% of the pupils were of ethnic minority origin. Pupils in this study experienced an emancipatory classroom. This involved pupils taking considerable responsibility for the direction and pace of the mathematics learning within the restrictions imposed externally. Open-ended mathematical activities were introduced as a whole class discussion with pupils and teacher making suggestions for possible routes for exploration. Most of the subsequent work was in small groups of two to six pupils, though the class was sometimes drawn together for a few minutes at various times in a lesson to enable a pupil to explain a discovery or for the teacher to raise a learning or organisational point that had arisen. Interaction between groups to share information or ideas was common. The teacher circulated amongst the groups, supporting directions of thinking, questioning directions of thinking, actively intervening to challenge directions of thinking, assisting decisions made towards solving a problem and responding to requests for help. Small group organisation was on a self-selection (usually friendship) basis. The management of small groups could be described as "low structure management" (Fawns and Sadler 1996); that is to say, there was no direct teaching of group skills for small group work. The construction of mathematical understanding as a joint endeavour and the self-selection of groups provided the impetus for pupils to develop 'norms' about necessary skills within groups.

Audio-data were collected from collaborative small groups comprising friends in naturalistic settings in mathematics classrooms during their normal activity of solving open-ended mathematical problems. These data were analysed using Mercer's (1995) model of three levels (*linguistic*, *psychological*, and *cultural*) to analyse peer discussion. The following extracts are analysed at the *psychological* level. This utilises an analysis of thought in action. It identifies to what extent reasoning is visible in the talk. It involves the communication structures between learners, the extent to which learners control the content and direction of the talk, and the 'ground rules' established for what constitutes valid talk within the group, what Cobb and Bauersfeld (1995) call 'sociomathematical norms'.

Two lenses on peer talk

The first extract is from a Year 8 group (12-13 year olds), discussing the problem of deciding a route through a series of rooms containing 'bags of gold' according to the number of the room. Hence, for the 2 x 4 grid which follows,



a route which omits room 2 will maximise the number of bags of gold collected. In the following extract, the group are discussing a 3 x 3 grid.

E This goes across … it goes one, two, three, four …
M It goes one, two, three, four,
E …five, six …
M seven, eight, nine …
J And I've noticed with this one, yeah …
E and it goes one, two, three, six, five, four …
J …you can do *all* of them …
E …seven, eight, nine
J …you can do *all* of them on this one
E all of what?
? [inaudible comment]
[Pause for 3 seconds]
M Maybe it's the way … maybe it's the way you set it out, though
E You can, actually, you can either go like that …
J …like that to reach every single one …
E …or you can go in … well, maybe not …
M If you go *down* like that, one, two, three, you won't be able to do the pink pen
E You would, would you …
J Yes, you would
K You could, yeah
J You could, with any of it, its just one, one more that you can do anything with

Using a socio-constructivist lens, we can see that J asserts in lines 5 and 7 that all bags of gold can be collected in a 3 x 3 grid, again in line 9, again in line 15, and finally in line 21. E, M and K eventually realise what J is asserting through counter-challenging M's challenges. This might be seen as evidence of learning taking place in Vygotsky's ZPD. Through a socio-cultural lens, the pupils are continuing each others' sentences, evidence that the group has established a cohesive and trusting community where the 'ground rules' require repeated repetition and confirmation of colleagues' ideas and opinions. Members of the group are talking aloud, to place their thoughts in the public domain (lines 13, 16 and 18), allowing the rest of the group access to these thoughts. J allows the rest of the group time to arrive at her level of understanding of the problem, through 'talking themselves into it'. This offers evidence of the mutual development of shared understandings of the situation.

The second extract is from a Year 9 group (13-14 year olds) who are exploring two sequences of numbers which have a logarithmic relationship. This task is an introduction to this mathematical knowledge.

R [giggles] I still don't understand
M Right, you know the log of the graph here … have you done C yet?
R Huh?
M Have you done C?

L This graph
R Yeah
L We're going to do the same but with the log numbers
R OK … you'll have logs of the C numbers at the bottom and logs of the values of the C numbers at the top. Then, what?
L Then it should be easy to see the relationship between the two
R And where you are going to put the places for each one, like
M It'll give you almost a straight line up to there
R Yeah, like … nought point three against whatever that one is
M Yeah
R OK
J How can that show the relationship?
L Cos it will be a straighter line
J Yeah, so if it's a straight line, you've got, um, C numbers … the log of C numbers here and its got the rank log, … number … this log
L The log value of the C numbers
J The log value of the C numbers … Oh right, I get it, …yeah
L It's a bit simpler
J Yeah, I know
[Pause as all work independently for 8 seconds]
J Yeah, I get it
M I know what it shows now
J You do eight times eight times eight times eight
L Pardon?
J Eight times eight times eight times eight
L Four times yeah?
J Yeah, OK yeah, … so … log … [whispers some numbers] … equals … four …right?
L Ummm? Yeah
J Oh, I'm so brilliant … OK, I can do this as well

Using a socio-constructivist lens, the way in which M and L support R and J's learning could be said to be acting in their ZPD. The quality of their explanations to aid R and J's understanding support Webb's (1991) findings about the level of mathematical learning in groups being directly proportional to the quality of explanations given by members of the group to each other. Both R's responses in lines 13, 18 and 19, and J's responses in lines 27, 29 31, 32 and 34 indicate a high level of understanding of the explanations given.

A socio-cultural lens shows us that this level of understanding is partially borne out of the equal mathematical status each of these group members share and as an outcome of the established (unwritten) 'rules' about ways of working together mathematically. Both R and J are unconcerned that, in this particular situation, they appear to be the weaker learners. Their confident questioning of M and L and their talking aloud to clarify their thinking and place their thoughts (whether they be right or wrong) in the public domain indicate that they are secure within the working practices of this group. This supports evidence from classroom studies of very much younger children who establish socio-mathematical norms when working together in groups. Although J does not enter this discussion until line 16, it is with a clearly directed question, demonstrating an understanding of the conversation which has ensued. She has been 'tied into' the reasoning because, as Wittgenstein (cited by Ernest 1998) argues, the knowledge is made public through the various types of talk – talking aloud, direct questioning, explaining, repetition of others. What is particularly interesting, in this extract of talk, is that the longer established friendship pairings of M and R, and J and L, appear to have given way to the learning needs of individuals in this group. All participants are actively engaged with the necessary thinking to move the group forward.

Discussion

The analyses provided here offer evidence of some of the affordances of group work in secondary mathematics classrooms and examples of how the dynamics and interactions of groups support the construction of a continuous (or shared) thinking space to support their mutual mathematical understanding. The socio-constructivist theoretical lens enables us to focus on transference of new knowledge from inter-individual to intra-individual through the ZPD. The socio-cultural lens allows us to examine Kinginger's "dialectic unity of learning-and-development" and explore the structures within the group via the use of language.

Here, in the 'threads of talk' linked together by repetitions or continuation of each other's sentences, we see evidence of what Mercer declared was essential for developing joint understanding - strongly established context and continuity. This is also evident in the way each of these groups have an established set of 'rules' or sociomathematical norms – a way of engaging with each other as equals, despite an apparent inequality of understanding in a given learning situation.

Such attention to learning theories in relation to classroom practice should be seen as an important focus, not only for teachers and researchers, but also for policy makers. The recent trends to 'roll out' yet another pedagogical practice, without due attention to the learning theories which support these, inevitably leads to the current situation in which teachers are expected to implement the changing whims of policy makers, currently a shift from the 'back-to-basics' to a focus on processes and skills. A focus on learning theories, rather than competing pedagogies, would go far to support teachers' practices in secondary mathematics classrooms and allow them some autonomy to develop a pedagogy appropriate to their individual situations.

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Lower secondary school students' knowledge of fractions

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In this paper we present some preliminary data from the ESRC funded ICCAMS project, and compare current Key Stage 3 students' performance on fractions and decimals items with students from 1977. We also present some interview data concerning students' models of fractions, and in particular their use of diagrams to represent part-whole relationships.

Keywords**: Fractions, decimals, models**

Background

Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) is a 4-year research project funded by the Economic and Social Research Council in the UK (Hodgen et al. 2008). In this paper, we report and discuss early findings of the study regarding students' understanding of fractions. In particular, we compare Key Stage 3 (ages 11-14) students' performance in 2008 and 1977 on items probing their understanding of fractions and decimals. We then discuss some interview responses to a fractions item, with particular reference to their use of diagrams.

Methods and theoretical framework

Phase 1 of the ICCAMS project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England using three CSMS tests, Algebra, Ratio and Decimals, and an attitudes questionnaire. Items from the Fractions test were added to the Ratio test in the 2008 administration. These tests were carefully designed over the 5-year project starting with diagnostic interviews. (See Hart 1981, for a discussion of the test development.) In Phase 2 of the study we are conducting a collaborative research study with eight teachers extending the investigation to classroom / group settings and examining how assessment can be used to improve attainment and attitudes.

The data in this paper are drawn from the Phase 1 Ratio and Decimals tests and from Phase 2 group interviews.

Participants

In June and July 2008, tests were administered to a sample of approximately 3000 students from 10 schools and approximately 90 classes. We report here on items from the Decimals test and the Ratio test (to which we had appended some fractions items). 2015 students took the Decimals test and 2022 students took the Ratio test. The sample was randomised and drawn from MidYIS, the Middle Years Information System. MidYIS is a value added reporting system provided by Durham University, which is widely used across England (Tymms and Coe 2003). When the cross-sectional survey is completed in 2009 with a further group of 3000 students, the sample will be representative of schools and students in England.

Theoretical framework

There seems to be a widespread consensus among researchers that the rational number construct can usefully be seen in terms of five subconstructs: part-whole relations, ratios, quotients, measures and operations (e.g., Kieren 1980; Behr et al. 1992; Pitkethly and Hunting 1996). Pitkethly and Hunting (1996), in their review of research into the development of early fraction concepts, go on to suggest that the part-whole and ratio subconstructs are fundamental in this development. Part-whole relations are strongly emphasised in the English National Curriculum, and some have argued that this emphasis may be to the detriment of a broader understanding of fraction. Kerslake (1986) suggests that by starting out from the part-whole model, "a major accommodation is required before a fraction can be thought of as a number or as the result of dividing the numerator by the denominator" (p 89). She further states:

> The ready availability of the 'part of a whole' model may itself be the inhibiting feature. If, in thinking of the fraction 3/4, say, the image that immediately springs to mind is that of a circle split into four parts of which three are shaded, then it may prove difficult to adjust to an alternative image of three circles and four people. (p 90)

Nunes (2006) makes a similar point and goes as far as to suggest that the division model (exemplified by, say, 6 children sharing 2 pizzas) chimes better with young children's intuitions about fractions.

Since the introduction of the National Curriculum in the 1980s, the number line also features strongly in English schools. It is used particularly for learning about operations on whole numbers in the primary school, and on integers in the secondary school. However, as we shall see, our data also suggest it has had a positive impact on the subconstruct of rational number as measure, at least as far as decimal representations are concerned (our interviews seem to suggest that this does not apply to common fractions, although we lack test data on this).

Early Test Analysis: Student performance on fractions and decimals

We note that our early test results should be treated with caution. In particular, we note that the survey is due to be completed in Summer 2009 and that our current sample of students appears to be slightly higher attaining than the general population in England. This early and at this stage tentative analysis suggests that, at age 14, attainment in decimals has risen, is largely unchanged in ratio and has fallen in fractions. The changes in relative performance in decimals and ratio is perhaps unsurprising in that it reflects changes in the balance of the primary and secondary mathematics curriculum. Moreover, the use of decimals generally is far more widespread now than 30 years ago. However, taken as a whole, the data suggest that the well-publicized increases in examination performance in England are not matched by increases in conceptual understanding across mathematics. We emphasize again that this is early analysis and a fuller and more detailed analysis will be published in due course. Further, we note that the patterns across the attainment range, across the age range and across items appear to be rather more complex. The items discussed below have been chosen to be illustrative of student progression and the differential performance of items.

Item 6d (Figure 1) is typical of the broad pattern of attainment in decimals. This item is designed to test students' conceptual understandings of decimal place value in relation to the number line. As can be seen graphically in Figure 1, the item facility has risen considerably from 1977 to 2008 across the 11-14 age range. For example, at age 14 (Year 9), the facility for this item in 2008 was 83% compared to 50% in 1977. Indeed, the current facility of 70% at age 12 (Year 7) is higher than that for age 14 in 1977. One explanation of this is the increased use of the number line in the primary mathematics curriculum (Askew et al. 2002).

Items 12e (Decimals) and F18 (Ratio, originally Fractions) illustrate the difference in performance within and between tests. These items ask *how many* fractions / numbers lie between $\frac{1}{2}$ and $\frac{1}{4}$ and 0.41 and 0.42, respectively. The facilities are shown graphically in Figure 2. As can be seen, there is an improvement in performance on the decimals item, but this is very slight (at Year 9, 23% in 2008 against 21% in 1977). This may be because the item is non-routine and could be said to involve an element of problem solving. Performance on the fractions item has declined (at Year 9, 6% in 2008 against 15% in 1977). This may be because there is now less emphasis on fractions (as opposed to decimals) in the curriculum.

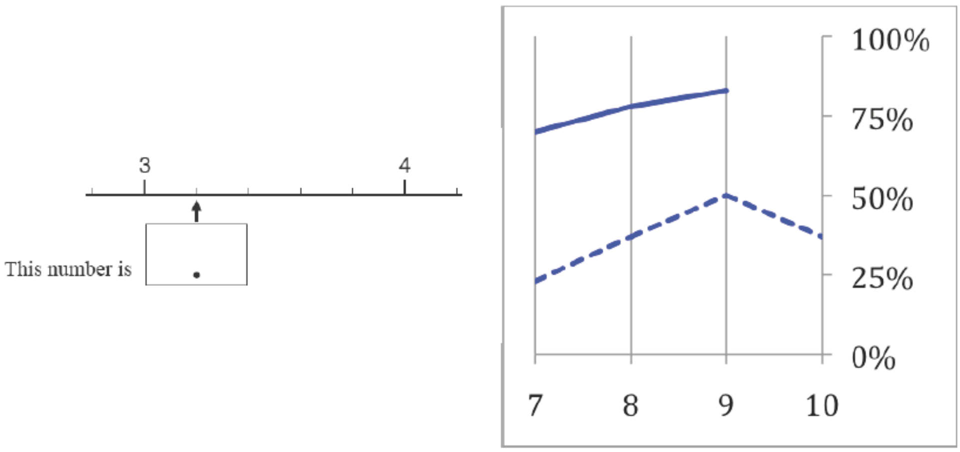

Figure 1: Decimals item 6d. The item is presented alongside several other similar items and students are asked to give their answers as decimals. Facilities are shown for the item in both 2008 [continuous] and 1977 [dotted] for Year 7 to Year 10 (ages 11-15). In 2008 data were not collected for Year 10.

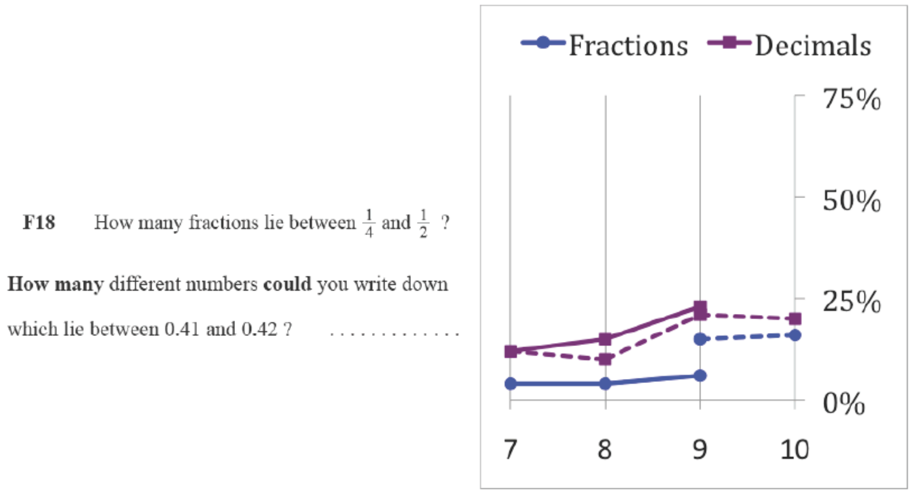

Figure 2: Fractions item 18 and Decimals item 12e. Facilities for items asking how many fractions / numbers lie between $\frac{1}{2}$ and $\frac{1}{4}$ [Fractions; Item 18], and 0.41 and 0.42 [Decimals: Item 12e]. Facilities are shown for both 2008 [continuous] and 1977 [dotted] for Year 7 to Year 10 (ages 11-15). In 2008 data were not collected for Year 10, whilst in 1977, data for the fractions item were collected only for Year 9 and Year 10.

Findings from interviews

We have used item F18 in several open-ended group interviews (of 2, 3 or 4 Year 8 students). The item gave us the opportunity to see what spontaneous models students had for fractions.

One quite common tendency was to think in terms of decimals. Thus one group of students had decided that 1/3 lay between 1/4 and 1/2. This was justified by one student in terms of ‘the bigger the number (denominator), the smaller the fraction’, while another gave this explanation: “A half is 0.5 and a third is 0.3 and a 4th is 0.25 and it’s in between 0.25 and 0.5”.

Not surprisingly, another common tendency was to use a part-whole model to represent fractions, usually by considering parts of a circle (or pizza, etc). A group of two students, R and T, had also decided that 1/3 lay between 1/4 and 1/2. T then suggested 1/5, which R rejected as being too small, because “if you got a circle and split it into quarters, if you split it into 5ths there’s one more to get in there”. This was a nice, grounded explanation, but interestingly involving an imagined rather than an actual drawing. T then suggested 3/5. Asked how we might check this, R suggested “Draw a pie”, which she proceeded to do quite effectively (see Figure 3, below). Using the diagram R was able to reject 3/5 and to decide that “2/5 would be OK”.

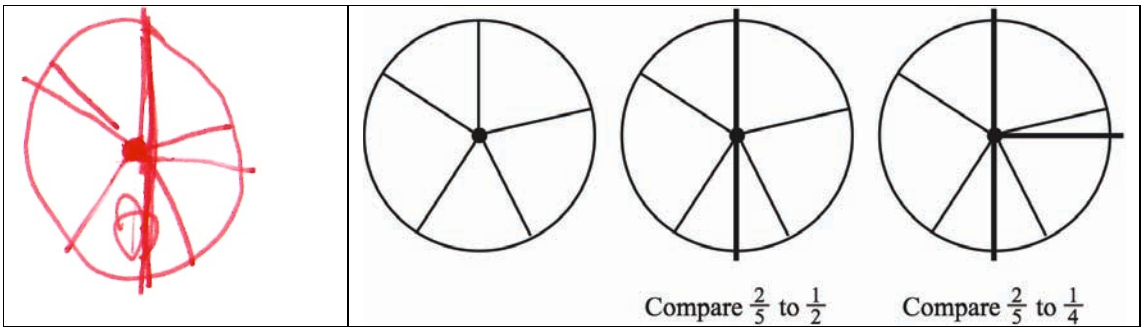

Figure 3. R’s diagram showing 5ths, used to compare 3/5 and 2/5 to 1/2 and 1/4

However, and somewhat to our surprise, R then suggested 3/6 as a possible fraction between 1/4 and 1/2. She proceeded to draw a circle to represent 6ths, which she did in quite a sophisticated way, by drawing diameters through the circle (Figure 4, below). This might be thought to suggest R had some intuitive understanding of how 6ths relate to 1/2, but strangely, she then halved the circle not by using one of her partition lines but by drawing a vertical line which passed through two of the regions representing 6ths. R somehow concluded that 3/6 is smaller than 1/2.

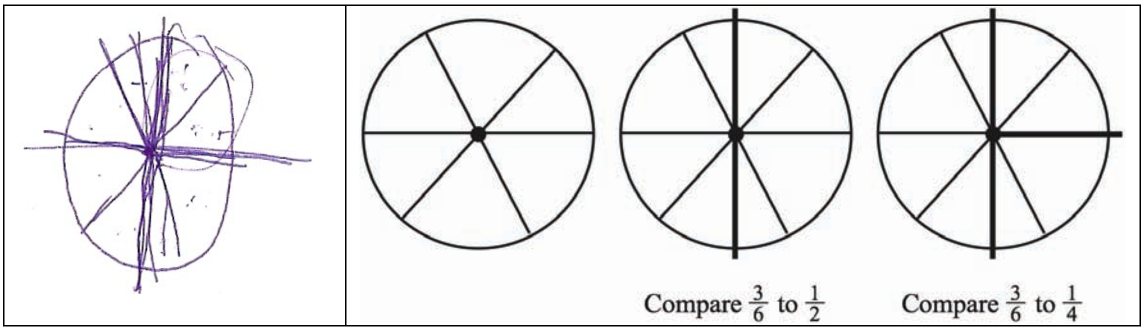

Figure 4. R’s diagram showing 6ths, used to compare 3/6 to 1/2 and 1/4

R then drew another circle to represent 6ths, but this time it was done in a rather short-sighted, step-by step way, resulting in quite irregular sized partitions (Figure 5, below), and leading to R abandoning the drawing.

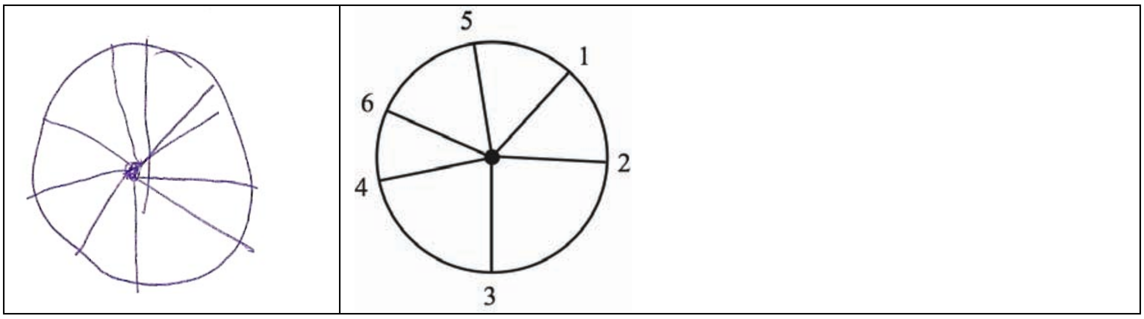

Figure 5. R's second diagram showing 6ths, with the order in which the lines were drawn

T was then asked what he thought and hesitantly replied "I think it would be the same as half, wouldn't it?", whereupon R exclaimed "Yes!". This interchange suggests that R had known that 1/2 and 3/6 are equivalent, but that her use of diagrams had not helped her retrieve this knowledge. There seems to be a paradox here. The diagram is being seen as providing concrete evidence but often it can only be used as an aid to thinking if it is *not* taken 'literally' but merely as a rough representation of an ideal. Put another way, for students to draw effective diagrams, they must be aware in advance of the relationships they are trying to represent.

Figure 6a

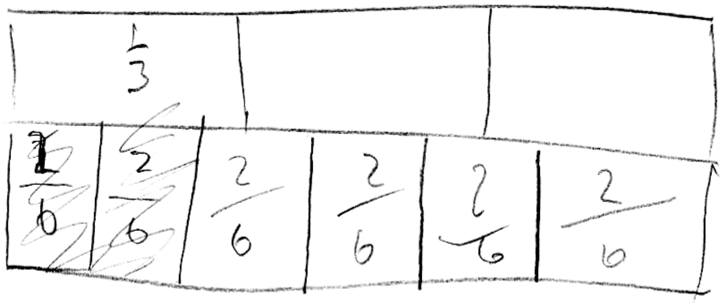

Figure 6b

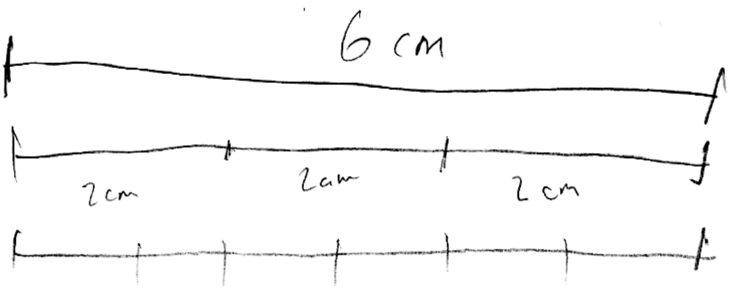

Figure 6. Fraction wall and number line to compare 1/3 to 2/6

With another group of three students we had got on to drawing a fraction wall to represent a whole, halves, 3rds, quarters, 5ths and 6ths. During the course of this one of the students concluded from their drawing that 2/6 was less than a third. We agreed that we needed to be cautious about concluding this as our drawing was not very accurate, and so the interviewer sketched a new wall to show 3rds and asked the students to draw 6ths underneath (Figure 6a, above). Unfortunately, it turned out that the resulting drawing confirmed their misconception. Again, the student who drew this, had proceeded in an empirical, step by step way, when what was needed was the realisation, in advance, that one could extend the partition lines for the thirds, and that two 6ths would fit into one 3rd.

The students suggested that we needed to use a ruler and in response to this the interviewer drew a line, notionally 6 cm long (Figure 6b, above), and a second similar line notionally divided into 2cm lengths to represent 3rds. The students were then asked to mark off 6ths on a third similar line, which this time they were able to do very effectively.

Thus, with the structuring offered by the idea of a ruler, and the conveniently chosen length of 6 cm, the students were this time able to show the equivalent fractions. These observations fit with those of other researchers. Thus, for example, Kerslake (1986) found that all of her interview sample of 12 - 14 year old students could read-off some equivalent fractions when shown ready-made diagrams with identical shaded regions partitioned in different ways. On the other hand, students would draw diagrams to confirm rather than to test their errors (e.g., to show that 3/4 is larger than 4/5, or that 2/3 + 3/4 = 5/7). A similar phenomenon is reported by Herman et al. (2004). This suggests that using one's own diagrams effectively is much more demanding, and more indicative of a sound understanding, than using ready-made diagrams.

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Linking Geometry and Algebra: English and Taiwanese Upper Secondary Teachers' Approaches to the use of GeoGebra

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The idea of the integration of dynamic geometry and computer algebra and the implementation of open-source software in mathematics teaching underpins new approaches to studying teachers' thinking and technological artefacts in use. This study opens by reviewing the evolving design of dynamic geometry and computer algebra; teachers' conceptions and pioneering uses of GeoGebra; and early sketches of GeoGebra mainstream use in teaching practices. This research has investigated English and Taiwanese upper-secondary teachers' attitudes and practices regarding GeoGebra. More specifically, it has sought to gain an understanding of the teachers' conceptions of technology and how their pedagogies incorporate dynamic manipulation with GeoGebra into mathematical discourse.

Keywords: Geometry; Algebra; Open Source Software; Comparative Study.

Introduction

Algebra and geometry are two core strands of mathematics curricula throughout the world and are considered the 'two formal pillars' of mathematics (Atiyah, 2001). It is therefore not surprising that they have been specifically targeted by the field of technology (Sangwin, 2007). Many researchers consider mathematics education as one of the earlier education fields to introduce technology as an assistant tool in classrooms (Papert, 1980; Noss and Hoyles, 1996).

The major application of technology in mathematics education is the integration of mathematical software in teaching practices. In respect of geometry, the most widely used computer applications, known as Dynamic Geometry Software (DGS) and include, Cabri-géomètre and Geometer's Sketchpad (GSP), etc. One important feature of DGS is the drag mode, encouraging interactions between teachers, students and mathematics (Jones, 2000). The drag mode can be used to explore and visualise geometrical properties by dragging objects and transforming figures in ways beyond the scope of traditional paper-and-pencil geometry (Laborde, 2001; Ruthven, 2005). DGS also has options to visualise the paths of objects as they move. For algebra, the most widely used applications are known as Computer Algebra Systems (CAS) and include programmes such as Mathematica, Maple and Derive. Some graphical visualisation and symbolic representations of algebraic expressions are implemented in CAS. Using the metaphor of the two 'formal pillars' of mathematics, geometry and algebra are afforded prominent positions especially at the secondary level (Hohenwarter & Jones, 2007). However, the connection between geometry and algebra, is apparently missing, as evident that in some countries geometry and algebra are entirely separate in their curricula (ibid). Although research into current technology use of computer algebra and dynamic geometry in teaching practices separate each sphere into distinct areas for study; I argue against this separation as there are areas overlapping algebra and geometry such as functions and graphs (Dubinsky and Harel, 1992). Examining both together has great educational implications and the connections between the two should not be ignored (Edwards & Jones, 2006). However, there is a gap in the literature dealing with this linkage between both fields and the use of technology. Despite an awareness of the need for a combination of DGS and CAS (Hohenwarter & Fush, 2004), software designers struggle to combine them as there are completely different constructs in software design. GeoGebra could be seen as pioneering software, although whether or not it is successful in linking DGS and CAS still needs research as the supporting evidence is limited at present.

Comparative Study

Recent research has indicated that culture influences the ways that teachers behave and inter-culture differences appears to be stronger than intra-culture differences (Schmidt et al., 1996; Givvin et al., 2005; Andrews, 2007). In particular, comparing eastern and western traditions with their respective Confucian and Socratic underpinnings can be enlightening as there are great differences in teacher beliefs and practices (Leung, 1995; Tweed and Lehman, 2002; Andrews, 2007). There is little comparative research of technology use in mathematics education, especially between Eastern Asian and Western countries (Graf. and Leung, 2001). Consequently, seeing how culture influences technology-mediated mathematics teaching is a pertinent issue.

There are large-scale quantitative studies such as TIMSS and PISA and small-scale qualitative studies. These studies highlight both similarities and differences between mathematics education in different cultural contexts in depth and in breadth. Quantitative studies such as TIMSS have also been reproached for their uncritical evaluation and for promoting globalisation over curricular and cultural diversity (Andrews, 2007). In contrast, small qualitative studies acknowledge cultural differences without attempts for generalisation. Particularly, when comparing East Asian and Western traditions with their respective Confucian and Socratic underpinnings, there is a significant difference between what are classically designed with the educational traditions (Leung, 1995; Kaiser et al., 2005; Tweed and Lehman, 2002). In particular, Kaiser et al. (2005) proposed a framework analysing East Asian and West European cultural traditions in mathematics education. The framework by Kaiser et al. (2005) is adapted partially in terms of teaching styles as I undertake a small-scale qualitative study in countries that exemplify East and West with a focus on teachers' perspective and their use of technology in mathematics teaching. The Eastern country chosen is Taiwan since it is viewed as 'the one most often cited admiringly by educators in the West for the level of its students' educational achievements (Broadfoot, et al., 2000: 13)' and a high mathematics performing country in international comparative studies such as TIMSS and PISA (Mullis, 2003; OECD, 2004; 2007). The Western country chosen for the study is England due to its contrasting educational system (Broadfoot et al., 2000). A cross-cultural study between Taiwan and England helps obtain a sense of the commonalities and discrepancies of teachers' conceptions and practices in relation to GeoGebra use. I have chosen to research at the upper-secondary level (students aged 15-18) as this level is less researched but is a crucial step for bridging students' secondary mathematics learning and higher education. Therefore, the overarching research questions are: (1)What are the upper-secondary mathematics teachers' conceptions of technology in relation to GeoGebra in England and Taiwan? (2) In what manner is GeoGebra used for the teaching of geometry and algebra by Taiwanese and English teachers? (3)How are the teachers' conceptions of technology and GeoGebra related to their teaching practices in both countries?

Methodology

Since there is little research into GeoGebra usage to date, this study is exploratory (Marshall and Rossman, 2006; Creswell, 2007). In brief, exploratory and multiple-case studies are my chosen methodology as the research focuses on this particular mathematical software, requiring specific teachers who utilise GeoGebra to teach upper-secondary level mathematics. Comparing and contrasting cases of teachers with interest in using GeoGebra from Taiwan and England provide a comprehensive understanding of how GeoGebra can be used in two very different cultural traditions, pedagogies and curricula.

I define mathematics teaching with the use of GeoGebra in Taiwan and England as the two main units of analysis. These have embedded two cases of teachers who use this software. Moreover, within the units, four cases of English and Taiwanese teachers are studied to obtain evidence of their views on GeoGebra teaching practices. To achieve the comparability between cases and units, pre-determined themes: teacher background, views on technology and GeoGebra, software comparisons and ways of using GeoGebra have been set for research design and data collection. A complete set of data was collected from four school visits. All of the interviews were audio and video-recorded, lasted for approximately an hour each and took place in classrooms using either a laptop or a computer connected to an interactive whiteboard. During the interviews the teachers demonstrated ways they utilised the software. The interview data were collated and summarised for each of the four cases. Two of the cases will be introduced in the following passage.

The cases

Li

Li has thirteen years of teaching experience at the upper-secondary level in Taiwan. Since his first degree was in applied mathematics, he gained an interest in IT during his undergraduate study. He was enthusiastic about new technologies and volunteered to translate the Traditional Chinese version of GeoGebra. Moreover, he had been creative in using different software packages, free software in particular, and trying to use a combination of different open-source software to make teaching materials. He has written some journal articles comparing new, open-source software packages detailing how they might be incorporated into mathematics teaching for Taiwanese teachers. In addition, he conducted GeoGebra training courses and workshops for teachers in senior high schools in Taipei. He had also set up his website and school website and uploaded his up-to-date GeoGebra materials and step-by-step tutorial materials for students or teachers. Li had a similar opinion to Jay on students' and teachers' attitudes towards the use of computers. However, he was positive that exploiting GeoGebra can change students' attitude towards mathematics. Some of his designed teaching materials and tutoring examples of using GeoGebra in solving examination problems were displayed on the websites. He also encouraged students to use the websites for reference and discussion. The salient categories are listed as:

Tool use	Graphing, calculations, demonstration, problem-solving, revision, investigation, and interaction
Mathematics topics	Geometrical topics and algebraic calculations
Teaching style	Curriculum-based, textbook-oriented and exam-driven, self-developed teaching materials and website with GeoGebra
Infrastructure	Home, IT room or computer and projector in classroom

Tyler

Tyler has taught mathematics to 11-16 year olds in a college for twelve years. He has also acted as an AST1 supporting schools and as a part-time school consultant, cooperated with the NCETM GeoGebra project and hosted a GeoGebra training workshop at his college. Tyler's utterances reflected a view of GeoGebra as an environment for exploring dynamic geometry rather than algebra. He viewed GeoGebra as a replacement to Cabri, which he used before GeoGebra. However, he mentioned that his experience with GeoGebra was approximately half a year, which meant that there were areas of using GeoGebra that were under-explored and underdeveloped, such as using GeoGebra in teaching algebra.

Some criticisms about current usage of technology in schools were brought up in terms of the IT rooms and school websites. He described his intention to change the way his pupils work from being passive to actively involve in learning through software. Moreover, he did not expect that students would not undertake much thinking in the IT room. In addition, some school mathematics websites have mathematics tests for pupils to log on to at home with their personal passwords which, in his view, allowed no room for discussion and interaction. He pointed out that GeoGebra is interactive and intuitive so he could set up diagrams and activities for students to interact with easily: 'This is different. This is maths by interacting; this is maths by trying things out, by conjecturing, by having a go.' He emphasised that GeoGebra could not only be used as a presentation tool by teachers but also as an investigation tool for pupils. An enthusiasm for GeoGebra was apparent in Tyler's strategies of using GeoGebra in mathematics teaching.

Overall, Tyler was reflective and explorative about different practices with GeoGebra, and eager to find out possible areas where GeoGebra could be useful in mathematics teaching. He also drew a distinction between 'knowing how' to use it and 'getting used to' using it in relation with GeoGebra. This inferred that he acknowledged the differences between using GeoGebra and teaching with the use of GeoGebra. The salient categories emerged from the data are listed as follows:

Tool use	Demonstration, interaction, investigation, exploration, testing hypothesis, creation, projection capability and the slider
Mathematics topics	Mainly geometrical topics
Teaching style	A whole-class teaching activity
Infrastructure	Home, IT room or computer and projector in classroom

Findings

Analysing the data thematically across the case studies revealed four salient dimensions in relation to GeoGebra-assisted teaching: **educational tools, teacher transition, mathematical scope and infrastructural change**. The findings are

[1] Advanced Skills Teacher

introduced in the following, which indicate that understanding the linkage between teachers' conceptions and practices is crucial. Firstly, the teachers' conceptions of GeoGebra seemed to be strongly rooted in their conceptions of the effectiveness and infrastructure of technology. The English teachers imbued a more positive attitude towards technology than their Taiwanese counterparts. However, teachers in both countries expressed favourable opinions regarding GeoGebra's agreeable contribution to their teaching. Secondly, GeoGebra was commonly used as a tool for visualisation, demonstration and interaction of mathematical topics, whereas for algebraic topics it was rarely utilised in England. It appeared that the English teachers associated GeoGebra primarily with geometric topics. Conversely, Taiwanese teachers worked with GeoGebra on both geometric and algebraic topics as they did not consider algebra and geometry to be necessarily separate; possibly as a result of the structure of Taiwanese curriculum and textbook-oriented culture. Thirdly, there were three different environments where teachers engaged with GeoGebra: - preparation of teaching materials at home, presentation and interaction in classrooms and activities for pupil investigation in IT rooms. Teacher transitions evolved from and were influenced by the infrastructure as they moved from preparation to presentation, incorporating interaction with pupils and finally encouraging investigation.

Conclusion

There are three aspects generated from the data that could be seen significantly different between the cultures in England and Taiwan. Firstly, teachers' attitudes towards technology in both countries varied. The participated Taiwanese teachers held negative conceptions of technology use for teaching practices, whereas the English teachers were positive about it not only because they were confident and comfortable about using ICT but also students seemed to have higher level of acceptance. Secondly, the Taiwanese teachers experienced greater difficulties pertaining to infrastructure as the classroom settings were not particularly designed for technology use in Taiwan whilst the English classroom settings implemented interactive whiteboards and projectors which offered convenience for teachers. Finally, in terms of pedagogy, the Taiwanese teachers tended to follow a curriculum based teaching strategy and mostly related GeoGebra exercises to textbooks; therefore, GeoGebra was used specifically for assistance of visualisation of textbooks examples. Again, the English teachers appeared to be more creative and flexible in choosing their teaching methods. As the Taiwanese educational system has an examination-driven culture, there are several areas being used extensively such as problem solving for university entrance examinations and proof of theorems as well as revision for examination preparation. In contrast with Taiwan, the English educational system has a focus on individual learning, therefore, there seemed to be a stress on students' individual investigation and interaction with GeoGebra.

Despite the potentiality of GeoGebra, teachers have not fully discovered its capability to link geometry and algebra but acknowledged that it offers pervading possibility in teaching practices. As Markus Hohenwarter puts it, 'GeoGebra is free software because I believe education should be free. This philosophy makes it easy to convince teachers to give this tool a try, even if they haven't used technology in their classrooms before'.

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A-level mathematics: a qualification for entry to quantitative university courses

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Meetings with concerned groups of academics with a particular interest in the mathematics knowledge of students when they arrive at university are reported. There was general agreement on two immediately tractable issues and the appropriate actions: an A-level mathematics curriculum without options, so as to maximize students' knowledge in common, and examinations that test understanding and not merely memory and manipulative skill- so as to encourage deeper learning than at present. The relatively low numbers taking A-level mathematics is a much tougher issue. The consequences include many university courses in quantitative subjects admitting students without A-level mathematics, and adapting content and teaching accordingly, so as to survive. The underlying problem is to understand the unpopularity of mathematics after GCSE and what might be done about it.

Keywords: Mathematics knowledge; Curriculum; A-level; Quantitative courses; Student numbers.

Rationale

A generation ago school maths was a route for socio-economic mobility. The author's grandfather was a railwayman, his father an inner-city secondary school maths teacher, and he himself became a university professor, and he knows others whose families made a similar journey over two or three generations. But this route no longer seems to be open- certainly not to the same extent. In fact it seems mathematics is generally unattractive to the present generation of secondary school students- despite the potential rewards.

The author has spent his career in science and technology in higher education, and is researching an alternative secondary school maths curriculum- in the hope that, by being more in tune with the needs of 21st century life, maths will become more attractive to a new generation of learners. A-level maths is an important interface- to university courses and hence to the professions. So, one aspect of the author's research involves identifying and focussing on the shortcomings of the maths curriculum at A-level, from this perspective.

Introduction

Since November the author has been visiting a range of university departments offering courses leading to various mathematics dependent professions and talking with admissions tutors and those academics with a special interest in students' mathematical knowledge. All the courses require significant mathematical knowledge- although to differing extents- and the universities occupy different prestige levels.

The sample of visited universities covers three different prestige levels:

Top-ranking- research dominated
Middle ranking
1992 (ex-Polytechnic).

The courses investigated were:

Business and Management
Economics,
Computer Science/Informatics,
Engineering,
Mathematics, and
Physical Sciences.

The aim of sampling in these two dimensions is to try to ensure that the full range of any A-level maths curriculum shortcomings and opportunities for improvement- at least as far as preparation for the professions is concerned- will be uncovered. Hopefully this sample coverage gave accurate insights across a full enough range of both subject and A-level achievement.

Discussions with academics

I had prepared a range of curriculum content questions for my meetings with the academics, including-

a. Is the maths curriculum content sufficiently forward looking, in general, for 21st century needs? For example- is there sufficient emphasis on discrete maths?

b. Is good practice in calculator usage, including emphasis on estimation, good preparation for use of the software packages used extensively in professions?

c. Relational databases are not covered at A-level and yet relational databases support most Internet applications- so they are both topical and of great practical importance- and the underlying relational algebra would surely be rewarding?

d. School mathematics uses reasoning solely for mathematical problems. Surely there is great scope to use mathematical notation, especially graphical notation, and mathematical reasoning in non-mathematical areas- for example reasoning using causal networks?

But my meetings with academics, mostly admissions tutors for HE courses with a strong mathematical content or application, turned out perhaps unsurprisingly to be dominated by their concerns, and these together with their underlying causes have since become mine too.

I encountered general dissatisfaction with the maths knowledge of students arriving at university. I wanted to know whether changes in curriculum content would improve matters significantly. The answer I received was a qualified “yes”. It was qualified because their main concern was depth of the students’ knowledge rather than specific content. I constantly heard that the maths knowledge students acquire at school is “shallow” and often is not retained after the examinations, and this is much more of an issue than the particular maths topics in the curriculum. I was curious about the causes. But an even bigger problem turned out to be the number of students studying maths to A-level, and therefore qualified to enter quantitative courses. There are fewer than there used to be, despite the expansion of universities generally, with destabilising consequences for university courses, particularly those in the middle and lower tier institutions.

So, the scope of my investigation has inevitably changed- from testing my ideas about introducing 21st Century information age content into the curriculum to aiming to understand more pressing concerns about the state of the maths knowledge interface to university courses. My initial questions are on the back burner for now.

Concerns about secondary mathematics education

The following is a distillation of what I learned from discussion with these academics together with my thinking about underlying causes. It is separated under the headings below for clarity of explanation but the topics are all interconnected.

1. Mathematics knowledge

There was agreement that Maths knowledge is for working on problems. Two kinds of knowledge, for two kinds of problem situation were recognised. *Shallow knowledge*- using particular problem solving skills in situations where knowledge is routinely matched to a class of standard problems. "Calculus for technicians" was an example quoted. *Deeper/coherent knowledge- r*ecognising what pieces of maths, that may have been taught/learned separately, are together relevant in a particular problem situation. Importantly the problem may have to be formulated before it can be solved

2. Curriculum

a. Content

The remark commonly made by colleagues in Maths education, that "the curriculum is too crowded" didn't resonate. But there was agreement that Breadth versus Depth or Coherence is an issue- and the knowledge students emerge from exams with was described as "shallow".

b. Deeper implies narrower

So trading breadth for depth of knowledge became a discussion issue. If the curriculum must have reduced coverage to allow room in the timetable for greater depth, which topics should go? Will the various university disciplines that use maths agree on what should be sacrificed? There was one hopeful sign- an engineer said he would "Trade calculus for better knowledge of algebra".

c. Options

There was general unhappiness with the variety of maths coverage students had experienced at school and universal support for a Single maths A-level syllabus *without options*. Universities want this so all the students they admit have been exposed to the same topics at A-level (of course some will have absorbed more than others and some will have taken Further Maths, but the common coverage will have been maximised).

3. Examinations

The exams define the curriculum and this is inevitable with high stakes testing. The academics accepted this statement (though they, unrealistically I thought, deplored it). I am not sure they recognised all its implications. They made two distinct criticisms of the current tests: they encourage shallow knowledge- "learning for the test" (which is soon forgotten)- and they fail to identify those students with really high mathematical ability. I suggest the underlying causes are as follows.

a. Competition for business between exam boards

This is likely to encourage grade inflation- 25% A grades now, and rising. It was suggested that a single exam board would be beneficial development.

b. The cost of testing

Exam Boards are under pressure to minimise the cost of testing. The economics of large scale testing favours electronic assisted marking.

c. e-testing- the conseqeunces

Electronic assisted marking favours short questions with short answers. (The extreme case is Multiple Choice Questions which can be marked completely automatically.) And a consequence of this is fragmentation of maths knowledge into facts and short procedures which can be tested and marked cheaply. This is exactly what the academics mean by “shallow knowledge”.

d. Economics of testing

Testing deeper knowledge requires longer questions and/or project work and is significantly more costly. We can reasonably conclude that the economics of the current testing regime bears some responsibility for the shallow and fragmented maths knowledge of students and without spending more money on testing this is not going to change.

4. Numbers taking A-level maths

a. Contraction of maths numbers

Universities have expanded but maths A-level numbers have shrunk. There were 80,000 candidates in the 1980s, only 56,000 now, although numbers are rising again by about 8% per annum (STEM 2008). Adrian Smith (Smith 2004) attributes the fall to AS-level being too hard when it was introduced and deterring many students from continuing maths to A2 level. But in any case, even assuming numbers continue to increase and over time get back to the earlier level, this contrasts with an expansion in many other subjects.

b. Consequences of low numbers

The shortage of A-level maths students has greatly affected some universities and some disciplines leading to instability in the numbers meeting entry requirements at middle rank universities (see next section).

5. Effect on universities of the shortfall in A-level maths numbers

Universities have been affected differently according to their ranking.

a. Top tier universities

These have had first call on the maths-qualified students and so have been relatively unaffected and have generally expanded their intakes somewhat (but with some difficulty in the case of chemistry and physics).

b. Middle and lower tier universities

These, however, have been strongly affected by the shortfall in all quantitative courses, as follows. Departments have been required to fill student places or lose staff and ultimately close. The general response has been to lower admission criteria- ultimately reducing the maths knowledge entry requirement from A-level pass to GCSE. Even so, some courses have closed and some departments have closed. In some cases accepting GCSE maths as an entry qualification has meant doing a lot of remedial maths in the first year of the course. (The problem of these students' weak maths is compounded by their having done no maths during their A-level years. In principle a FSMQ is the answer here but, so far, there is not a sufficiently widespread take-up to allow admission tutors to insist on this. In any case they may feel they have insufficient influence over students' choices to do so.) Other courses have modified course focus- becoming more qualitative (Informatics courses, for example, have tended to do this, although some have closed).

6. What factors determine maths numbers?

a. Career choice

Maths knowledge can open so many career doors. So why don't more students take A-level maths? What career advice are students getting at school?

b. Dislike of maths

Do so many really dislike maths so much?

c. Maths is hard

This seems to be many students' perception and Adrian Smith (ibid) suggests that the GCSE exam in maths actually is harder than in other subjects.

d. Isolation of maths

Has demathematisation of the science curriculum contributed? Science and maths used to be mutually supportive.

e. Shortage of maths teachers and access to the curriculum

There is a shortage of maths teachers, particularly in some parts of the country, and some maths teachers are not very well qualified mathematically, in contrast with the subject knowledge of teachers in many other subjects. Does this effectively restrict the access of some students to maths? The maths teacher shortage seems likely to continue, or even get worse as older teachers retire. Can current maths A-level numbers be maintained/increased? If more students should want to take maths, could the schools actually accommodate them?

f. Access to Further Maths A-level

The Further Maths Network is helping mathematically ambitious A-level students to prepare for this A-level, despite the restrictions on classroom access to maths teaching at this level. Since FMN began, numbers at AS level have more than doubled to 8,000 and are up 50% at A2 (FMN 2008). This seems like a much needed ray of hope among so much gloom!

Conclusions

Modernising curriculum content is not the most urgent A-level maths reform. Two significant reforms that appear to be urgent and relatively straightforward to implement:

A single maths A-level syllabus with no options (ACME 2009) and, ideally, a single exam board, so as to maximise the maths knowledge in common of students entering university, and A-level exams that test deeper knowledge- a necessary precondition for encouraging students to acquire deeper knowledge.

It is to be hoped that both will happen soon, because there are other urgent and much less straightforward, issues as follows.

Maths A-level numbers have declined over time and by doing so have damaged universities' ability to offer courses in quantitative subjects, in contrast with their general expansion. Numbers are increasing again but have not yet returned to historic levels. For whatever reasons, maths A-level is not a popular choice. Students may not be getting good post-GCSE advice about the importance of maths for entry to so many careers. Students' choices can be dangerous for themselves and for universities!

A further concern is that teacher shortage threatens universal access to the maths curriculum, and could constrain any increase in numbers that might otherwise occur.

Future work

The author plans a further round of discussions with his university contacts, with the following objectives:

To coordinate their response in support of the ACME recommendation that there should be a Single Maths A-level Syllabus with no options, so that all students admitted to university have been exposed to the same A-level Maths curriculum;

To seek consensus on the content of the hoped-for Single Maths A-level Syllabus- what topics should covered and to what depth- and what should be left out- perhaps by ordering a priority list.

To seek a consensus on the importance of deeper assessment of knowledge at A-level- so as to encourage students' deeper understanding, while recognising that this will increase the cost of assessment.

He also intends to consider mechanisms for getting better post-GCSE advice to students about the career importance of taking A-level maths- what career doors may open for them that will otherwise stay shut.

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The Validation of a Semantic Model for the Interpretation of Mathematics in an Applied Mathematics Problem

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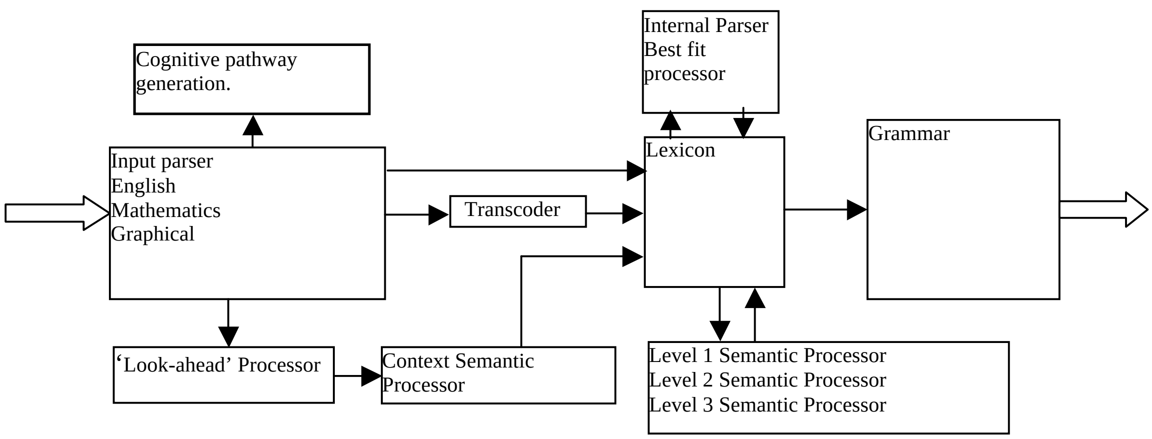
The semantic model proposed by Peters (2008) was developed whilst working with learners of mathematics solving algebraic problems. In order to investigate in more detail the role of the parsing process and its relationship to the lexicon, a different set of questions were devised based on Laurillard's (2002) work with undergraduate students. These same questions were also given to a set of mathematics tutors so that a comparison could be made between the two groups and to see if their behaviour could be explained using the semantic model. The analysis of these sets of data indeed show the importance of the parsing process and as predicted by the model, a competent mathematician employs a top-down parsing strategy.

Keywords**: Parsing, Lexicon, Cognitive Pathway.**

Introduction

The algebraic problems from the Chelsea Diagnostic Test (CDT) used in research that led to the development of a semantic model of the processing of mathematics (Peters, 2008) were all situated in a mathematical context and were abstract. A problem Laurillard (2002) was presented to undergraduate students to validate the semantic model and investigate the role of the parsing process and its relationship with the lexicon; it could be represented in a form of words, as a diagram and be interpreted using experiences of 'everyday' life. There were two scenarios: (1) a box resting on a table and, (2) a box in mid-air. Students were asked to explain the scenarios using Newton's Third Law of Motion. For the purpose of this investigation this problem was broken down into four separate questions; this required the learner to parse the orthographic form, the graphical form, access their lexicons and retrieve the appropriate entry.

The Semantic Model



The figure shows the semantic model developed by Peters (2008). This model was based on ones developed for arithmetic (e.g. McCloskey et al 1985), natural language processing (Chomsky 1981) and work by the author on how learners parse mathematical structures. It models the processes a learner of mathematics uses to interpret and make sense of mathematical problems. The process starts at the left where the learner reads the problem. The input parser is used in the initial stage to analyse the syntax of the problem and start the process of deriving the semantic content. If the problem is in a mathematical form i.e mathematical equations or expressions, the lexicon can be accessed directly but, if it uses a graph or natural language to set out the problem, a process of transcoding is used to alter the input into a form suitable to access the lexicon. The output from the lexicon works with the grammar to select the appropriate rules to resolve the problem. The initial parse of the problem also highlights the need to scan ahead and determine the context of the sign and ensure an appropriate semantic interpretation is attached. The cognitive pathway generation is the process where, if the problem is a familiar one, the relationship between the lexicon and grammar is known and a reduction in cognitive load can be achieved. If the form of the problem has not been encountered before a cognitive pathway has to be initiated. This pathway takes the form of developing the links between the cognitive modules. The semantic processing is depicted with three levels which arbitrary in the sense of a problem that was difficult to solve initially by the learner requires careful semantic interpretation whereas once the learner is confident with such problems the amount of semantic processing is reduced.

Research Methodology

The learners and the tutors were presented with the questions in a written format and in a specific order: vague to more detailed. Once a question had been presented and the answers given, the scripts were collected and the participants interviewed. This method was adopted so the more detailed formulation of later questions would not influence their responses to previous answers when interviewed. The interviews were conducted in pairs so that the discussion between both participants could be observed and analysed. The interviews were recorded and later transcribed for future analysis.

Validation of the Semantic Model

Question A: Explain how Newton's Laws of Motion apply to the two situations shown in Figure 2.

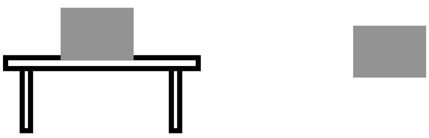


Figure 2, (a) A box resting on a table. (b) A box in mid-air.

This question was deliberately vague in respect to which law(s) of motion could be applied.

The written and verbal answers of the learners indicated that the initial parse of the question created problems. Learners who were unable to recall the laws indicated a failure between the parser and the lexicon. Their parsing of the question failed to identify the appropriate lexical entry, since the information gathered in the parsing process did not provide them with the necessary key information to facilitate the link between the parser and the lexicon. One learner's explanation, for which he did not state any assumptions, was that the box in mid-air must have been filled with a gas in order for it to float. Other learners stated assumptions such as; ignoring air resistance, the box was falling, model the box as a particle. In their written explanations all learners stated the box would fall due to gravity. None of them mentioned the force of the box on the Earth, but the notion of equal and opposite forces was mentioned in conjunction with the box resting on the table.

When asked what they interpreted to be the key words, the answers included '*explain', 'Newton's laws'.* One learner in the interview stated that once she had read the question her main worry was trying to remember Newton's laws of motion. She said that she thought she should have interpreted it in an 'everyday' context but did not do this because *"I was totally fixated on the maths, because that's what we've been doing."*

When asked if they could recall or if they had a vague idea of what they wanted, one of the learners responded with

> "yes I had a vague idea of what I wanted and I was cross with myself that I didn't know the topic well enough to be specific, because we've done it relatively recently."

One learner decided the third law was appropriate and gave a good explanation of the relationship between the forces in the box resting on the table. When it came to the box in mid-air, he could not identify a similar relationship. He knew gravity was 'pulling' the box towards the Earth and for the law to apply, he needed to be able to identify another force which he was unable to do. In the end he decided that air resistance provided the other force even though in his assumptions he opted to ignore air resistance.

Question B: Explain how Newton's Third Law of Motion applies to the two situations shown in figure 2. This question gave the learners more information; it told them specifically to use the third law which should have acted as a trigger for their lexicons. One of the learners underlined 'Third Law of Motion' in the question and in her explanation wrote "*Box on table exerting same force down as table exerts up ?3rd law."* In the interview, when asked her immediate thoughts, responded with "*What is the third law of motion. Is it the one I described in the last piece of paper and I don't know."* This response indicated they recognised the importance of the words (the key), '3rd law of motion', but a strong cognitive link had not be formed between the parsed words and the lexicon. Similarly one of the learners in the other pair responded with *"What's Newton's third law and that scared me, because I just didn't really know."* He went on to say:

> "...although it's the same question and I've already been influenced by question sheet A...but because it said third law, it's just like well I don't even know if I know the third law...and then I was thinking well, if the table was removed would the box just stay there, is it the same box, in which case the table is irrelevant..."

This learner also stated that once he read 'the 3rd law', he started panicking. When asked to explain what effect this panicking had on him, he stated:

> "...that's why I went into like freefall, that's when I started thinking about the table and things like that, cause it takes my mind off my stumbling block and I'm just trying to see if I can get around, there's another pathway to the answer that I want and I couldn't find it."

Question C: Explain how Newton's Third Law of Motion which states: 'every force has an equal and opposite reaction', applies to the two situations shown in figure 1.

This question gave learners a prepared statement of the third law in its canonical form.

The learners' written answers to this question included words such as 'force, reaction, equilibrium, at rest, forces cancel' whereas in question A and B they used 'force, equal, opposite, gravity'. This change in vocabulary indicated that they did have lexical entries for the more mathematical terms used in question C. The problem with the wording of the third law is that it omits any reference to the fact that the definition of the law requires that there exist two bodies exerting forces of equal magnitude on each other. As Laurillard (2002) pointed out the phrase 'equal and opposite' implies that the forces cancel out to give equilibrium which gives rise to the learners' misconception about the third law.

One learner gave a succinct answer: "Gravity pulls box down. Table pushes box up. Forces cancel and box stays still." Another answer was: "Normal reaction of the table balances the weight of the box." Their answers to scenario (b), the box in mid air, remained in essence the same as their previous answers except that the word 'force' was now used. For example, one learner stated:

> "If box is stationary must be some sort of upwards force to counter balance box force downwards otherwise this will not last long in mid-air."

Another responded with: "Gravity pulls box down. Force inside box pushing up – assume lighter than air gas. Forces cancel and box stays still.

One learner attempted to answer the two possible situations: the box moving towards the Earth and the box 'floating' in mid-air. His answers were:

> "Gravity is the only force acting on the box' and 'if the box is in mid-air (floating) then lift should equal mg."

Another learner in an attempt to resolve the point of ambiguity resulting from her interpretation of diagram (b) suggested:

> "If we assume the box is in mid-air is in a vacuum it will have no forces acting on it so will stay where it is."

Question D: Explain how Newton's Third Law of Motion (When one object exerts a force on a second object, the second object exerts an equal and opposite force on the first), applies to the two situations shown in figure 1. The problem with this question for the learners was the word 'object'. In part (a) they could explicitly see the objects involved whereas in part (b) only one is shown. Their answers, both written and oral, supported the view that there was only one object present in part (b) i.e. the box.

One learner in the interview stated in response to being asked if the question was different:

> "Yes very different, because you put the word objects in. So you said one object is exerting a force on a second object and going back to my poor box in mid-air it has no other objects..."

The other learner in this pair correctly identified the Earth as the other object but this just created a point of ambiguity.

> "...I got confused and couldn't decide whether it was relating to this law of motion or another law of motion which would account for situations...as you were saying (referring to other interviewee) the Earth is a factor that should be taken into consideration...I'm trying to trawl through all the laws and try and work out that there's other things that are equal and opposite but they don't happen to have objects in them."

In their written explanations one of them wrote: ‘Hadn’t considered objects so important but I suppose other motion laws account for other situations’. The other wrote ‘...still not happy with the box in mid-air, there’s no second object...’. The notion that the Earth can be considered as an object seemed difficult for them to grasp. Two of the learners confirmed this when asked what they considered to be the problematic words:

> “B: The key word is object, different word...makes you think in a different way.
>
> A: I think you immediately think of an object as something...(B interrupts).
>
> B: Something tangible, three dimensional you can see and feel...it was almost superfluous because when we looked at question C we almost assumed that...we did not really need to be told they were objects as well...and we’re thinking of the Earth and gravity and somehow you don’t think of them as objects in the same way (referring to question D).”

One of the learners had difficulty in the overall structure of the formulation of the third law. When asked about the question in general, he replied:

> “Too wordy. I struggled to read it, too many objects within a sentence and Newton’s third law the way it was explained in the question was just too wordy and it took me too long to try and work out what the third law was, even though I already knew from the previous question.”

Analysis of Tutor Answers

Initially the tutors could not see the point of the question since it was obvious to them how the scenario should be resolved. This indicated that their lexical entry for the situation was very well developed. Although they parsed the question, the diagram was used as the main source of obtaining the necessary information.

In response to question B:

> “…But I can’t recall which one we call the first and which one we call the second…I don’t think, to be honest, I don’t think it really makes any difference when you come to analyse a problem as long as you know those two concepts.”

The tutors did not question the box in mid-air; they immediately assumed the diagram was a ‘snap-shot’ and consequently the box was accelerating towards the Earth; they did not debate how the box could be suspended in mid-air. This was clarified in response to question C when asked what assumptions they made:

> “Well, if it’s in mid-air, it’s gotta move. If it’s in a gravitational field of any kind.”

It seems as a part of their parsing process, assumptions were automatically generated and any unrealistic situation was immediately discounted. They assumed implicitly that the scenario was Earth based and a gravitational force existed.

Summary

The responses from the learners who participated in this exercise highlight the difficulties that arise from underdeveloped lexicons and the parsing process. In terms of my semantic model it seems that when they parsed question A the cognitive pathways were initialised. Question A was not specific and did not mention the third law and learners used their existing conceptions. As the questions became more specific, these conceptions were challenged resulting in ambiguity which needed to be resolved. When asked, if they were in an exam, which question they would be most comfortable with, they replied question C. The formulation of the third law in this question was probably one they were familiar with and therefore they considered they understood it. The questions also highlight the difficulty learners have relating everyday experience with the mathematical interpretation of phenomena. It was apparent from this study that once the learners had reinforced the cognitive pathways initially set on parsing question A (reinforced by question B since it did not challenge their conception of the third law) they had difficulty in resolving the scenario in a mathematical context. It seems that they attempted to make the mathematics fit their conceptions rather than reparse and reset their cognitive pathways.

This study also highlighted the importance of wording questions in terms that are understood by the learner. For example, in this study it was assumed the learners would know what was meant by ‘object’ in question D and that they knew that gravity was a force. This particular problem concerning the nature of gravity might have been due to the fact the force has been named, unlike ‘general’ forces, and therefore had become reified in the lexicons of the learners. If the lexical entry was missing the learners adopted a ‘best-fit’ approach; they attempted to analyse the problem using ‘folk definitions’ and use their everyday experience to explain the scenarios.

Once the lexical entries are well defined the learners can begin to progress from bottom-up parsing to a more efficient top-down parsing strategy. Once they are able to use a top-down parsing approach, facilitated by the combining of ‘small’ concepts to form ‘super’ concepts, learners understand the semantics of the question.

If the learners’ conceptions are compared to the tutors, it is apparent that the bottom-up parsing approach adopted by the learners created difficulties. They tended to focus upon the atomic structures of the questions and in a way lost sight of the problem. On the other hand the tutors, who were very familiar with this type of contrived problem, parsed the questions in a top-down fashion and hence did not focus on the wording of the question. They were in some respects at a loss to see what the problem was; to them the solution was obvious.

It seems from this validation process that when a problem is parsed any preconceived notions, including assumptions, are linked to the lexical entry. In the case above, the tutors naturally assumed the box was in a gravitational field and therefore could not remain suspended in mid-air. The learners did not make this assumption automatically and therefore spent time trying to justify how the box could remain in mid-air. The implications of this on their cognitive load are enormous. If the learner has to spend time and hence cognitive resources trying to find a resolution that is unreasonable then it is no wonder they become frustrated and rely upon ‘folk definitions’ of mathematical concepts.

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Mathematical Knowledge in Teaching: the Nuffield seminar series

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Over the last two years, BSRLM members from several universities have contributed to a national seminar series on Mathematical Knowledge in Teaching that has met on six occasions. The final report of the series, supported by the Nuffield Foundation, is now available, and an edited book is in preparation. The seminars have examined current scholarship and research bearing on how teachers' subject-related knowledge underpins successful mathematics teaching, and on how such knowledge can be assessed and developed. As a consequence, it has been possible to identify areas where there is a need for further research in this important field.

Keywords: mathematics teaching, teacher knowledge, seminar series.

Introduction

The purpose of this paper is to report the activity and findings of a seminar series on Mathematical Knowledge in Teaching between January 2007 and June 2008. The six meetings, over seven days, were funded by the Nuffield Foundation. Further details of the six meetings, including a number of papers presented, can be accessed from http://www.mkit.maths-ed.org.uk/

The Nuffield seminar series

Intellectual aims and organisation

The overarching intellectual aim of this seminar series was to draw together current ideas and evidence about the forms and functions of the mathematically-related knowledge which enables teachers to support successful student learning of mathematics. More specific aims have been to achieve a critical conceptual synthesis, to establish significant professional implications, and to identify major research needs. Thus, the seminar series consisted of an opening 2-day conference followed by five 1-day conferences, each with a specific focus. Three of these events were concerned with taking a critical overview of existing thinking and research on the key issues of conceptualising and theorising mathematical knowledge for teaching, auditing and assessing such knowledge, and developing and deepening such knowledge. Two further events were concerned with closer analysis of a selection of research studies and teaching resources related to substantive aspects of mathematical knowledge for teaching. The specific aspects were division and fractions, and argumentation and proof, chosen because they are well-researched and contrasting aspects of mathematics addressed at multiple levels of education. The final event focused on formulating a research agenda. An Appendix to this paper provides a summary overview of contributions to the seminar programme on each of these themes; in particular, of the papers and presentations prepared by speakers.

To encourage a focus on critical synthesis and professional implications, a specification was provided for presenting speakers (in a fairly standard form, adapted to fit the focus of the particular meeting), emphasising the importance of speakers explaining how and why ideas or methods represent a significant advance on, or alternative to, earlier ones, and identifying significant implications of bringing these ideas or methods to bear on the practices of teaching, and of teacher education and development. Similarly, the specification enjoined group discussions to identify commonalities and contrasts, complementarities and conflicts between different ideas and methods; and to identify any significant limitations of the ideas and methods in illuminating important practical issues, and any significant limitations of current policy and practice in acknowledging important insights from these ideas and methods.

Professional aims and organisation

The overarching professional aim of this seminar series was to build research capacity in this area. More specific aims were to establish a working network covering teacher educators and educational researchers, but also extending to potential research user groups with a direct professional interest in the area; and to provide particular opportunities for doctoral student researchers working in this area to participate in the network. Thus the network that has been established includes mathematics education researchers and teacher educators at various career stages (including doctoral students). Recognising that future teachers develop much of their mathematical knowledge at school and university, prior to entering courses specifically devoted to teacher education, we recruited some persons active in designing and teaching undergraduate (specialist and service) mathematics courses to join the core membership. To involve practitioner and policymaker communities concerned with mathematics teacher education and training, we were fortunate to be able to recruit as core members of the seminar series participants representing the Department for Education and Skills [DfES, now DCSF/DIUS], the National Centre for Excellence in the Teaching of Mathematics [NCETM], and the Office for Standards in Education [OfStEd]. Although our attempts to recruit school-based subject mentors to core membership of the series proved unsuccessful, at each of the university-hosted meetings we were able to attract such colleagues from amongst those associated with local teacher education activities. Likewise, other doctoral students and academic staff of the host institution joined these seminars as local attendees. Finally, a number of other interested persons, primarily mathematics education researchers and teacher educators visiting from overseas, attended particular seminars.

Findings

The main findings from the conceptual synthesis which has taken place over the course of the seminar series can be related to the research needs and professional implications identified at the final meeting.

One important need is for research on mathematical knowledge in teaching to take on a more programmatic character. This has a number of aspects. First, research to date has tended to focus on particular phases, topics and processes: on primary teachers and teaching rather than secondary or tertiary; on arithmetic rather than algebra, geometry or probability and statistics; on proof and proving rather than on models and modelling. Second, a range of approaches, drawing on different sources of evidence, have been developed in order to identify what mathematical knowledge is important for teaching, but insufficient attention has been given to triangulating – and potentially integrating – these approaches and understanding the relationships between them. A more systematic research programme would provide a stronger knowledge base for designing courses of initial teacher education and providing professional development for serving teachers.

A further need is for research which probes some of the assumptions which have pervaded the work undertaken to date, and explores the viability of alternative conceptualisations of mathematical knowledge in teaching and corresponding ways of investigating its operation. For example, much work to date has conceived mathematical knowledge in teaching as strictly individual knowledge capable of being elicited in isolation from the actual teaching situation, and as independent of the tools and resources available to support subject teaching. Equally, most research, even when it has compared educational systems, has proceeded on the assumption that mathematical knowledge in teaching is independent of particular contexts for (and cultures of) mathematics education. Finally, the field has tended to emphasise those aspects of mathematical knowledge which are specific to teaching, perhaps at the expense of more generic aspects.

A final need is for a stronger emphasis on research aimed at developing and validating tools to support the enhancement of mathematical knowledge for teaching within the initial and continuing professional education of teachers; research and tools locating such knowledge more strongly within everyday processes of teaching, and relating it more directly to providing effective support for student learning of mathematics. Examples of such tools examined during the seminar series include diagnostic 'mathsmaps' to help teachers reflect on, and develop, their mathematical knowledge; 'lesson study' as a vehicle for collaborative development of mathematical knowledge for teaching in the context of preservice teacher education; and the 'knowledge quartet', an analytic framework for the identification and discussion of primary teachers' mathematical knowledge as evidenced in their teaching.

The last stage of the seminar series coincided with publication of the interim and final reports of the Williams review of primary mathematics teaching (raising many issues equally applicable to mathematics teaching in other phases). During the final seminar, several papers examined the main recommendations of the Williams review, endorsing the main recommendation that professional development for teachers should focus on "the three interrelated strands of mathematical content, mathematical pedagogy and embedded practice", but also noting some of the challenges of implementing the review's proposals. The review's emphasis on more informal modes of teacher development through in-school mentoring and coaching calls for investigation of how best to embed professional learning around teachers' everyday work in school communities in ways which enhance mathematical knowledge, for example through forms of lesson-focused peer interaction. Equally, the review's emphasis on peer interaction may underestimate the potential contribution of what have been termed 'educative' curriculum materials, specially designed to support the mathematical learning of teachers as well as their students.

Dissemination

Three main avenues of dissemination from the seminar series are being pursued. First, an open website was established at the start of the project (http://www.mkit.maths-ed.org.uk) to make public details of the series, and provide access to documents prepared in the course of it. Now that the seminar series has been completed, this website provides a valuable and readily accessible archive of its work and findings.

Second, to further develop the synthesis undertaken by the seminar series, and make it more widely available, an edited book is now in preparation. We expect this to appear in 2010 in the Springer *Mathematics Education Library*. The main chapters will develop papers and presentations given during the seminar series. The editorial guidelines emphasise the same concern as the seminar specifications that authors should make clear what intellectual progress and professional implications are represented by the research that they are reviewing and presenting. This represents a valuable opportunity to refine the work undertaken during the seminar series, and to further strengthen critical synthesis, particularly through the inclusion of discussion chapters at the end of each section of the book which will draw on the seminar discussions to explicitly address that brief.

Third, we made two presentations at the February 2009 day-conference of the *British Society for Research into Learning Mathematics*. The first of these focused on conceptions of Mathematical Knowledge in Teaching, with contributions from Marilena Petrou, Maria Goulding, Jeremy Hodgen, Anne Watson, Andreas Stylianides and Jill Adler. In this session, the speakers presented snapshots of the different theoretical and practical perspectives examined in the seminar series. In the second session, Julie Ryan, Dolores Corcoran and Fay Turner reported on empirical research into the use of particular theorised tools in the development of Mathematical Knowledge in Teaching, involving pre-service and early careers teachers. Ken Ruthven concluded by outlining some areas where the need for further research was identified.

Appendix

Brief details of the six seminars are as follows.

Seminar 1: Conceptualising and theorising mathematical knowledge in teaching

January 2007, Cambridge. http://www.mkit.maths-ed.org.uk/seminar1.html

The seminar was structured around the following invited presentations:

Maria Goulding - Mathematical subject knowledge in primary teacher training: a view from England and Wales.

Jeremy Hodgen - The situated nature of mathematics teacher knowledge.

Heinz Steinbring - Changed views on mathematical knowledge in the course of didactical theory development.

Dina Tirosh & Ruhama Even - Teachers' knowledge of students' mathematical learning: an examination of a commonly held assumption.

Kenneth Ruthven – Synthesis.

Seminar 3: Auditing and assessing mathematical knowledge in teaching

September 2007, London. http://www.mkit.maths-ed.org.uk/seminar3.html

The seminar was structured around the following invited presentations:

Julian Williams - Audit and evaluation of pedagogy: towards a sociocultural perspective.

Tim Rowland - Auditing the mathematics subject matter knowledge of elementary school teachers.

Julie Ryan & Julian Williams - Mathsmaps for diagnostic assessment with pre-service teachers.

Marilena Petrou - Michigan research on developing a practice-based theory of content knowledge of teaching.

Seminar 5: Developing and deepening mathematical knowledge in teaching

March 2008, Loughborough. http://www.mkit.maths-ed.org.uk/seminar5.html

The seminar was structured around the following invited presentations:

Anne Watson - Developing and deepening mathematical knowledge in teaching: being and knowing.

Fay Turner & Tim Rowland - The Knowledge Quartet: a means of developing and deepening Mathematical Knowledge in Teaching.

Birgit Pepin & Linda Haggarty - Making connections and seeking understanding: mathematical tasks in English, French and German textbooks.

Seminars 2 and 4: Mathematical knowledge in teaching: examining the case of division and fractions (Seminar 2) and the case of argumentation and proof (Seminar 4)

April 2007, Manchester. http://www.mkit.maths-ed.org.uk/seminar2.html

January 2008, Cambridge. http://www.mkit.maths-ed.org.uk/seminar4.html

At these meetings, discussion of research studies and teaching resources was stimulated by critical reflections on published research related to the above substantive topics. These were prepared and presented by:

Dolores Corcoran, Johannes Siemons, Ray Huntley, Lara Alcock, Peter Huckstep and Sandy Pepperell (the case of division and fractions);

Maria Goulding, Marie Joubert, Andreas Stylianides, Cathy Smith and Johannes Siemons (the case of argumentation and proof).

Seminar 6: Formulating a research agenda on mathematical knowledge in teaching

June 2008, London. http://www.mkit.maths-ed.org.uk/seminar5.html

At this final meeting of the series, several core members of the seminar made brief presentations, each with a supporting paper. These presentations were organised thematically as follows.

Towards a programmatic framework

Maria Goulding & Marilena Petrou - Conceptualising teachers' mathematical knowledge for teaching.

Andreas Stylianides - Towards a research programme for identifying what mathematical knowledge is important for teaching.

Johannes Siemons - Mathematics knowledge in teaching: Formulating research.

Julian Williams - Towards a conceptualisation of a 'collective teachers' mathematical knowledge.

Towards more informed provision

Lara Alcock - The relative impact and teacher perceptions of different kinds of professional development.

Julie Ryan & Julian Williams - Teachers' stories of mathematical knowledge.

Ray Huntley & Peter Huckstep - The place of exemplification in mathematical knowledge.

Anne Watson - How can learning more maths impact on teaching?

Towards a comparative perspective

Paul Andrews - The cultural location of teachers' mathematical knowledge: another hidden variable in research on mathematical knowledge for teaching?

Birgit Pepin - What kinds of knowledge help teachers to become effective teachers of mathematics? What kinds of choices do teachers have? – a comparative perspective.

Towards a broadened agenda

Dolores Corcoran & Sandy Pepperell - Mathematical knowledge in teaching for social justice.

Marie Joubert - Using ICT in mathematics classrooms.

Into the post-Williams era

Tim Rowland & Fay Turner - Research into how deep knowledge of mathematics may be developed through ITE and PPD.

Kenneth Ruthven - The need for a programme of research on educative curriculum materials as a mechanism for the diffusion of mathematical knowledge in and for teaching.

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Online Reference

The Nuffield Seminar Series in Mathematical Knowledge in Teaching (MKiT) http://www.mkit.maths-ed.org.uk/

ENRICHing Mathematics: Progress in Building a Problem-Solving Community

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The SHINE enriching mathematics project recruited secondary school students in two socio-economically deprived London boroughs for out-of-school workshops over the course of a year. Students worked collaboratively on open maths tasks, with discussion guided by NRICH leaders and participating school teachers. Here we outline two aspects of the project evaluation: how we analysed progress in collaborative classwork and how the students described what they had learnt. Students found Shine maths enjoyable, different and more challenging than school maths. Their teachers observed improvement in problem-solving behaviours. The model of a maths-talk learning community offered ways to categorise changing classroom behaviours, and helped to identify tensions and effective practices of classroom management.

BSRLM Keywords: problem-solving, enrichment, assessment

The Shine Project and Evaluation.

This research concerns the five-year "Shine" maths enrichment project, based in two London boroughs, and initially commissioned by a national charity, Shine. Secondary school students from years 8 and 10 in local schools met at weekly out-of-school workshops over a school year, and worked collaboratively on tasks drawn from the Nrich bank of problems. One of us, Jennifer Piggott, was involved in designing and running the project, with the challenge of creating a setting and pedagogy which supported students' mathematical engagement with problems. The other, Cathy Smith, designed and implemented an evaluation of three early cohorts. As the project developed the evaluation fed formatively into project design, and we have since collaborated to report and reflect on 'Shine' as an example of collaboration between schools, LEA and enrichment providers (Smith and Piggott 2007).

The aims of the project were to raise attainment in the areas of problem solving and mathematical thinking, and to raise students' aspirations and awareness of mathematics. Two year 10 cohorts and one year 8 cohort were studied in the evaluation; each with 35 to 50 participants recruited from 5-7 local schools. The project schedule varied for each cohort, involving between 45 and 58 hours of workshops. The Shine recruitment criteria focused on problem-solving potential but in practice schools recruited willing students from their higher sets. This was not a very narrow restriction: students achieved in the top 30% nationally, expecting from A* to C at maths GCSE, and Levels 6 to 8 in KS3 SATS. Recruitment and attendance were ongoing concerns involving careful liaison with schools: recruiting 70 students from 7 schools gave 50 long-term participants whose average attendance was 67%. The participating students were representative of the major ethnic groups in the two boroughs. The high proportion of Black and Asian students (above 70%) was unusual for enrichment projects and resulted from the school-level organisation.

Shine workshops lasted 2 or 3 hours and were based on a short starter problem and one or two longer tasks, taken from a schedule which involved revisiting and making connections between related tasks. Students worked individually, in groups, and in whole-class discussion. Sessions were led by Nrich staff or schoolteachers trained in the initial phase of the project. Leaders aimed to "create an atmosphere in which they engage in dialogue and other interactions including the use of modelling and metacognition and the use of props or cues, as teaching and learning tools"(Piggott 2007, p38). Spoken maths was the main focus of public attention and development, while written working was informal and private.

The evaluation considered possible impacts on student attitudes and attainment and what features of practice contributed to them. Its design was structured by balancing two concerns: to select attributes and use instruments that were compatible with the project pedagogy, and to collect attainment data relevant for school maths and for problem solving. A range of instruments was used to focus on these different aspects:

- before-and-after student questionnaires concerning attitude, enjoyment and aspirations;
- before-and-after teacher profiling of students on 12 problem-solving descriptors;
- attainment data from SATS and GCSEs, for participants and for 'matched' students identified by class teachers at the outset;
- teacher and student interviews;
- videos and observations during workshops.

Changes in the student cohorts over time and missing teacher data meant that the comparison sets had to be carefully identified, and necessarily excluded the experiences of students who left, and who had certain teachers. These quantitative analyses are seen as one perspective to inform the qualitative evaluation; they do not generalise straightforwardly. For school maths, GCSE results were higher than for matched students by an average 0.3 of a grade (paired sign test, 1%; data for 66 out of 85 students) and Yr 9 SATS were higher by 0.2 of a level (paired sign test, 5%; data for 34 out of 38 students). For problem solving, teacher profiles showed significant improvements in nearly half the attributes (Wilcoxon signed rank test, 5%; data for 79 out of 123 students). The greatest improvements were in students' ability to explain their reasoning, their interpretation and use of diagrams, and their formulation and manipulation in algebra. Details, and reasons for caution, are discussed further in Smith (2007). This paper considers two aspects of the evaluation, the analytic framework of a maths-talk learning community and students' reflective comments, to show how features of workshop practice contributed to beliefs on maths.

Observations

In the Nrich teaching style leaders stressed communal acts of speaking and recording, and managed transitions beween these and private episodes of individual thinking and group work. Rather than make individual performances visible, as would be the case with assessed tasks, the evaluation needed to analyse the workshops in terms of the balances and interactions between activity and passivity, public and private work, teacher and student. The framework used for this was the model for a maths–talk learning community developed by Hufferd-Ackles et al. Fuson and M. Sherin (2004) to research change during the US 'math reform' program. Classroom behaviour is described in terms of progress through levels 0 to 3 in each of the four inter-related areas of questioning, explaining, source of ideas and responsibility for learning.

Broadly, level 0 describes classrooms in which the teacher supplies and controls the mathematics attempted and discussed. Level 3 describes classrooms where students initiate ideas and extend each other's reasoning. At intermediate levels, teachers direct students' attention to each other's maths-talk.

Videos and field notes from 12 workshops were analysed to find examples of episodes of behaviour characteristic of each level, and then referenced to whether the observation occurred in the first half, ie 'early', or 'late' in each project. This provided a structure for producing descriptions of behaviour and change in each of the four framework areas. We discuss these below, summarising the level descriptors and exemplifying how we drew links between the supporting evidence and the features of practice that contributed to change.

Questioning

At level 1, teachers use follow-up questions that probe students' methods and thinking. Their phrasing models mathematical language and values.

> But is that all the solutions? Any other possibilities to explore? What do you think? Have we covered the whole field of possibilities there – are you convinced? (Obs, Y10)

Here the leader is directive but doesn't leave time for students to respond or to ask their own questions. The public questions are intended as a model for private questions that could guide individuals' future activities and metacognitive strategies. At level 2/3, teachers ask students to comment directly on each other's contribution.

> That's brilliant! Are you hearing this? This isn't about adding up - this is about understanding. Maria, can you hear OK? Just catch on to what Melody's saying… (Obs, Y8)

These questions focus attention on the activities of understanding and listening to others rather than on task aims. They direct the social space of the classroom, stopping just short of asking Maria to comment on Melody's work.

All the workshops showed level 1 questioning; some had level 2 questions inviting students to describe others' work or compare it with their own; in a few later workshops students were asked to critique others' reasoning but this was only achieved with very directive management techniques: short questions, instructions and names. Some students felt uneasy in this atmosphere:

> Teachers can be less pushy. I sometimes found it a bit intimidating especially when I didn't know an answer (Y8 questionnaire)

No workshops matched level 3 descriptors because students did not initiate questions about others' reasoning in public. There were however examples of increased questioning of others in small group work.

Explaining

At level 1, teachers elicit students' explanations but often restate and fill them out. In the following example the leader initially depersonalises the process of explaining, suggesting that 'maths' does the 'telling', but then links it back to student action when Jodi's explanation comes to a halt and she needs a prompt. Such details - the pragmatics of maths talk (Rowland 1999) - allow leaders, consciously or not, to manage student self-esteem by attributing ownership flexibly.

> T: It's not right . Can it tell us anything?
> Jodi: the left side needs 2 more to get 11, the right side needs 2 less. So…
> T: So which can you swap? (Obs, Y10)

At level 3 students expect that all results have to be explained. In later workshops this was clearly an understood classroom convention. For example Chaz and Iping were cautious (or lazy) when publicly contributing to a Sudoku problem:

> Chaz types in her numbers on the Product Sudoku screen.
> Iping calls out "You've got a lot of explaining to do".
> Chaz: "OMG you mean I've got to explain it all!", enters a few more - thoughtfully - deletes some - and returns to group (Obs notes, Y10).

This area showed the clearest change. In later sessions, leaders treated explanations as objects of mathematics available to be revised and critiqued. They might prompt students to re-state their own explanations several times in one interaction. They did not necessarily resolve students' incomplete or contradictory reasoning at the end of each whole-class phase. These practices helped to locate the source of maths ideas.

Source of ideas

In this area, there was little variation of level but some tensions in achieving it. Level 2 describes student ideas as the focus of classroom attention, with the teacher managing their interaction. The Shine workshops were designed at this level so leaders had clear guidelines that their role was to elicit student ideas and hold back from suggesting strategies. In early workshops there were episodes of unease when progress was slow and neither leaders nor students would volunteer ideas. Students who dropped out at that time described sessions as boring. Two related features that linked individual work to public work were useful in reducing these episodes. Leaders used group work time to circulate and rehearse students in ideas and explanations, allowing them to call publicly on those students later. One leader described this as a more purposeful version of his usual school practice. It was also made easier by the number of helpers present: usually 4 or 5 accompanying teachers and volunteer university students. Students found the potential attention liberating:

> Different groups have different ideas, and the teachers help with the different ideas. And if there's one teacher then you'd only be able to help with one idea and not like everyone's (interview Y10).

So, although there was often more help available than students used, it had a practical and symbolic role in creating space for multiple opinions, students as well as teachers.

Responsibility for learning

This fourth area concerns the connections that students make between their own learning and the public practices of the maths-talk community. At level 1, students are passive listeners and reporters, co-operating with teacher instructions. At levels 2/3 they also show that they expect to understand mathematics and, if they don't, they initiate talk about their understanding rather than seek instructions about what to do. Shine students sought help in relatively private interactions so whole-class observations showed no clear progress beyond level 1. In interviews, however, several students said that Shine maths involved more detail and persistence than school and linked this to trying repeatedly to understand:

> We look at problems and we take them apart and we try to explain every single bit carefully along the way. We try to find another way of getting the answer until

> we understand completely what the question is about, all the possible answers. (int Y8)

Even a student who was less aligned with this convention described it as part of the maths - doing the 'whole problem' – not just the requirement of an arbitrary teacher:

> You have to do the whole problem - it feels like a never-ending tunnel. I've got this bit and I can go on – now he says what about this bit! (int Y8)

In the first two areas, leaders did change the focus of their questioning and explaining, and students changed too. There were tensions when teachers directed students in unfamiliar practices but the changes were accepted, perhaps because they were recognisably linked to progress and authority. In the last two areas, the framework was used to show fixed level 2 classroom behaviour as far as this could be controlled by teachers. However, managing this level made demands for student input that conflicted with expectations about pace and social roles. Questioning and explaining were again important techniques for communicating the expectations. We observed that leaders blurred distinctions between private and public knowledge, and whether demands came from teachers or from the task, and suggest that this helped to resolve the paradox of instructing students to take responsibility.

Students' views

The observation framework raised the question of how students might independently take on ownership of mathematics ideas and responsibility for learning. The evaluation operationalised this as students' self-assessment on progress within Shine and its transferability to their school maths, described via scored statements and open questions in questionnaires (84 paired plus 32 initial and 16 final) and interviews.

Over 90% of students felt that they had improved in all aspects of problem-solving maths. Over 80% also felt the project had helped their school maths, but only moderately and with no consensus as to how. Year 10s had the most extreme reactions either way: several who dropped out explained that it wouldn't help them at all with school maths, but some articulated benefits clearly:

> "Yes because at first in most exams, most questions I rush to do it, but this time I take time and I think of different ways to do it. When I am stuck I think of the ways I do here." (Int, Y10)

There were however a few significant changes to questions about school maths. Year 8s believed less that answers were 'right or wrong'; and year 10 students questioned whether 'you do well in maths by copying your teacher'. Both of these open the way for contributing ideas and pursuing understanding in the classroom. In addition the proportion of pupils in each cohort who said that they enjoyed school maths lessons rose after the project by 9-15%. This counters the national trend that enjoyment of maths decreases with age even as students increase in confidence and attainment (Sturman and Twist 2004).

The students drew clear distinctions between the type of maths they worked on at Shine and what they did at school. Around 80% found it 'quite a lot or a lot' different, and 60-70% 'quite a lot or a lot' harder. They commented on the increased demands to contribute and explain:

> "We are pushed more to join in, it's not book work at all, the teachers encourage you loads to answer problems" (Y8)

> "It is harder and focuses on why not what" (Y8)

Analysing student comments showed three main sets: two that constructed the project as an addition to school maths, and one as a challenge. One set focused on having learnt a single skill described either as *problem-solving* or *working systematically*. Another focussed on extending a problem-solving repertoire, stressing the variety of possible approaches: *"I learnt different methods in solving"* (Y10). The last set described Shine as introducing a new perception of mathematics:

> "It's like two different whole subjects […] that's very similar, not just the one whole maths being taught in different ways." (Y8)

Pleasurable excitement about this 'new' maths was more common in the Year 8 cohort than Year 10. The notion of 'two' maths was followed up in interviews. Students reconciled differences by predicting that school maths would become like Shine when they progressed to GCSE/ A-level. This related to the perceived intensity of the problems but also to their application in context:

> "Yeah in schools we look at normal maths, symmetry or anything like that … Here we look at overall, world-wide. Like - the cinema problem – we don't do this stuff in school. It's based on what we do everyday - everyday stuff." (Y10)

Problems often use contrived contexts in which to create mathematics, eg forming algebraic expressions for cinema ticket prices. Perhaps the opportunity to stay in them for an extended time while developing a strategy is as important as the superficial realism, and is missing in school maths.

Finding benefits for school maths was one motivation for students, but we were also interested in whether the project offered intrinsic motivation that could sustain involvement through the unease noted in observations. Students were asked to describe their best achievement on the project. Responses were split between a sense of achievement about being involved in difficult tasks - "*Just the fact that I'm coming every week ever since I started and got on with the problem and not gave up"* - and the social enjoyment of *"solving problems with my friends and contributing to it*".

Shine set out to create a teaching environment that approached a maths-talk learning community. The evaluation confirmed that students became more ready and able to explain their own reasoning and attend to a range of strategies, and this contributed to success in solving problems. The practices used by teachers to promote public reasoning created some social tensions for students and teachers, and strategies emerged to reduce these. Students found the project challenging but motivating, with more differences than similarities to school maths.

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Developing the Ability to Respond to the Unexpected

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In this paper I present some findings from a four-year study into the development of content knowledge in beginning teachers using the Knowledge Quartet as a framework for reflection and discussion on the mathematical content of teaching. Findings which relate to the participants' ability to react to pupils' unexpected responses are discussed. Data from three case studies suggest that the framework helped participants to consider their unplanned actions when teaching mathematics. There was also evidence that over the course of the study the participants become more able to act contingently in relation to the mathematical content of their teaching.

Introduction

The basis of this study was the use by beginning teachers of a framework for the identification and development of mathematical content knowledge. This Knowledge Quartet (KQ) framework was developed by a group at the University of Cambridge (Rowland, 2008). The work of the Cambridge group focused on the classification of situations in which mathematical knowledge surfaces in teaching. The KQ offers this classification of the situations through which mathematical content knowledge of teachers is 'made visible' as a framework for the analysis of teaching. The KQ framework was developed from observation and videotaping of mathematics teaching. Analysis of this teaching produced 18 'emergent' codes (Glaser and Strauss, 1967) of situations in which mathematical content knowledge of teachers was 'made visible' e.g. concentration on procedures, making connections between concepts. These were later classified into four 'superordinate' categories based on associations between the original codes. These categories make up the four dimensions of the Knowledge Quartet; foundation, transformation, connection and contingency.

In this paper I focus on the development of teachers' content knowledge as 'made visible' through the lens of one of these categories or dimensions, that of situations in which teachers act contingently. The contingency dimension of the quartet may be considered to be about the ability to react to unplanned situations or to 'think on one's feet'. There are three codes which emerged from the empirical research subsumed under this category; deviation from agenda; responding to children's ideas and use of unplanned opportunities. Most mathematics lessons are planned before the act of teaching takes place and teachers bring their curriculum knowledge, subject matter knowledge (SMK) and pedagogical content knowledge (PCK) (Shulman, 1986) to the planning of a text for the lesson (Shulman, 1987). Teachers predict how pupils will respond to their planned teaching based on knowledge of content and students (Ball, Thames & Phelps, 2008) as well as on their previous experience of teaching, and amend their text accordingly. However, not all pupil responses can be predicted.

A teacher's ability to react appropriately to unplanned-for responses depends, at least in part, on their bank of SMK and PCK. Bishop (2001, pp. 95-96) offered an example of a teaching incident which illustrates the role of teacher's content knowledge in reacting to pupils' responses. In this example 9- 10-year-olds were asked to give a fraction between ½ and ¾. A response given was 2/3, "because 2 is between the 1 and the 3, and on the bottom the 3 lies between the 2 and the 4". The way in which a teacher might react to this response would depend on their SMK – were they aware of Farey sequences and mediants, and on their PCK – did they know how to disprove the generalisation inherent in the pupil's justification. The contingency dimension of the Knowledge Quartet therefore offers a lens through which to identify teachers' mathematical content knowledge.

The study

The aim of this study was to investigate the way in which beginning teachers' mathematical content knowledge for teaching might be developed through focused reflection using the KQ framework. Throughout the study this framework was used as a tool for analysis, evaluation and development of the teachers' mathematics content knowledge. The study began with 12 student teachers from the 2004/5 cohort of primary (5-11 years) postgraduate pre-service teacher education course at the University of Cambridge reducing, as anticipated, to 4 in the fourth and last year of the study. Data came from observation and analysis of teaching using the KQ as well as from post-lesson reflective interviews, group and individual interviews and participant written accounts. Transcripts of interviews and written reflective accounts were all systematically coded using the qualitative data analysis software NVivo. A grounded theory approach (Glaser and Strauss, 1967) was used which led to the emergence of a hierarchical organisation of codes into a number of themes. For a more detailed account of the research methodology see Turner (2008).

Case studies were built from the KQ analysis of observed teaching as well as from analysis of the data coded using NVivo. The analysis of observed teaching, using the 18 codes and four dimensions of the KQ, provided a 'spine' for presenting findings in relation to the development of participants' mathematical content knowledge. Data from the NVivo coding of interview and written data support, supplement and enrich these findings. Six themes in the development of the participants' mathematics teaching emerged from the NVivo coding. These were, beliefs, confidence, subject knowledge, experience, reflection and working with others. Discussion of findings about participants' mathematical content knowledge, including that revealed through their ability to act contingently, drew mainly on data from the NVivo themes of subject knowledge and confidence

Findings in relation to the ability to act contingently

Knowledge of errors and what they suggest about children's understanding of mathematical ideas is part of a teacher's PCK. The way teachers respond contingently to mathematical errors therefore gives some insight into their PCK. At the beginning of their teaching careers the participants did not always make good use of opportunities for teaching offered by children's errors. For example, during Amy's lesson observed during her training year in 2004/5, the class of 4-5 year old children were asked to write some 'teen' numbers. Amy focused on correcting children's reversals of digits but did not address their errors which involved writing the numerals in the wrong order e.g. '01' for ten and '21' for twelve. However, when reflecting on this lesson using the KQ framework Amy acknowledged that the ordering of the numerals was a more significant error and suggested that she should have used this to discuss place value rather than focusing on digit reversals.

The contingency dimension of the KQ framework helped Amy to think about a more useful way of responding to the children's errors. Such reflection may be described as reflection on action (Shön, 1983). The following year, Amy demonstrated the ability to respond helpfully to children's errors in action. At the conclusion of a lesson on counting some children were having difficulty counting the number of times Amy hit a chime bar, and continued counting after the last chime. Amy responded to this difficulty by asking the children to close their eyes, count the number of chimes in their heads and only give the answer once she had finished. Amy's knowledge of the cardinal principle enabled her to 'think on her feet' and suggested this effective strategy.

In a lesson observed later in 2005/6, Kate made good use of a child's error. Kate displayed a measuring cylinder on the interactive whiteboard and asked a volunteer to indicate the level to which 100ml of liquid would come. A child pointed to the interval marked '1000 ml'. Kate asked the class how they knew this did not show 100 millilitres and the children responded that it had an extra zero and was a thousand. Kate was clearly aware that children's errors can be used to advantage when teaching.

> I took advantage of Lily confusing 100 with 1000 on the interactive measuring cylinder to discuss place value. (Kate, reflective account of observed lesson, 2005/6)

Kate's use of the contingency dimension of the KQ helped her to think about how she responded to children's errors. She became increasingly confident in using children's errors to inform her teaching and appeared to relish such opportunities.

> When estimating how many cubes long a book was Harriet-Mae said "eighty" and then corrected herself to say "eighteen". I used this as an example to question the children about which of these was a sensible estimate and we discussed why 80 was not. (Kate, reflective account, 2006/7)

The use of appropriate resources in order to address unexpected difficulties is another aspect of acting contingently which was found to have developed over the course of the study, particularly in the case of Kate. An instance in which Kate made use of a resource, i.e.100 grids that she had not planned for was recorded in a reflective account of her mathematics teaching and demonstrates her concern with the flexible and appropriate use of resources.

> When we were suggesting different ways to count, the children wanted to count in 100s. Someone said 0, 100, 1000. I used 100 grids to represent units of 100 and then counted in 100s to 900. I asked the children if they knew another word for 10 hundred and someone said '110' so I had to demonstrate the difference using 10 hundred grids compared with 1 hundred grid and 10 cubes. (Kate, reflective account, 2006/7)

Jess was also aware of the need to be able to use resources contingently. Under the heading of 'Contingency', Jess recounted an incident in which she had been unable to use a particular resource to demonstrate why a child's response was incorrect.

> We were looking at lines of symmetry on the interactive whiteboard. The children were shown a variety of shapes and had to identify where the lines were and how many lines. When a child identified a line which didn't exist, I found it

> hard to prove they were wrong without actually folding paper. (Jess, reflective account, 2006/7)

Jess felt able to demonstrate that this answer was incorrect by using a different resource. However, more secure pedagogical content knowledge might have enabled her to make use of the interactive whiteboard.

Kate demonstrated secure PCK through her contingent use of a resource during a lesson observed in 2007/8. The lesson had been planned by another teacher as a Powerpoint presentation and Kate had not had the opportunity to make any amendments before teaching. Kate acted contingently when a slide was displayed showing 23 + 12 as (20 +3) + (3 + 2). This modeled a 10-10 strategy which she thought inappropriate. Kate 'deviated from the agenda' and made a new slide to model the N10 strategy i.e. 23 + 10 + 2. For a discussion of the 10-10 and N10 strategies see Beishuizen (2001). Kate's knowledge of the two methods enabled her to make an 'informed' choice about which to use. Kate further demonstrated her ability to use unplanned resources by producing a 100 grid and modeling the procedure for addition of ten by moving down one row.

Discussion of children's unexpected methods for solving problems was another form of contingent action observed during the study. Participants became more likely to carry out such discussion in their teaching over the course of the study. In the lesson I observed in her training year, Jess revised how to interpret pie charts with her class of 9-11year olds. Jess displayed a pie chart showing preferred flavours of ice cream and asked a question which involved calculating ¼ of 32. The class had had already found that 3/8 of 32 = 12 and 1/8 of 32 = 4. One child explained that he had calculated ¼ of 32 by adding 12 and 4 and dividing by two. There were rich opportunities for discussion of equivalent fractions in this but Jess simply responded "that works well". Our discussion in the post-lesson interview suggested that Jess did not see an opportunity for discussing equivalent fractions because she did not understand the child's method.

In the lesson observed early in 2006/7, Jess appeared more willing to explore a child's calculation method. Jess asked the children to record their methods for solving '20 ÷ 2' on individual whiteboards. Most children drew pictorial representations modelling the partative method that Jess had previously demonstrated. One child however, had simply written '20 ÷ 2 = 10'. When Jess asked how he had arrived at the answer he explained that he knew "ten add ten is twenty" so had "put ten on one side and ten on the other". Jess said "knowing ten add ten is twenty to find twenty divided by two is like using the opposite". Jess understood and related his method to division as the inverse of multiplication.

Over the course of the study, the participants became more likely to discuss children's methods of calculation. They were also more likely to ask for and accept children's ideas as starting points for their teaching. Amy recognised that she had missed an opportunity to work from children's ideas during the lesson observed in early 2005/6.

> The children in my 'treasure counting' group had some good ideas for how we could count all the coins more quickly. It would have been good to try out the children's ideas, despite asking for their suggestions I went ahead with what I had planned to teach them (Amy, reflective account, 2005/6).

Amy wrote this comment under the heading of 'Contingency', indicating that this dimension of the KQ had helped her to focus on how her teaching might encompass children's ideas. Comments in Jess' reflective accounts under the heading of 'Contingency' suggest that the KQ framework encouraged her to think about exploring children's thinking in order to make her teaching more meaningful.

> They often catch me out when discussing subtractions – "Why can't you do 4 – 8?" "You can, it's a minus number!" I have started to get these children to explain in more detail what they have said so I understand where they are coming from and also so some of the other children start to realise some of these things too. (Jess, reflective account, 2006/7)

During the study the participants became more willing to discuss children's methods and to explore their ideas. This was underpinned by a growing confidence that their mathematical content knowledge would enable them to understand the children's thinking and develop this in their teaching. An extension of this willingness to explore children's methods and ideas was a growing confidence to allow children to investigate mathematical ideas for themselves. In a lesson on measurement that I observed early in 2006/7, Amy gave the children a selection of objects and containers which they could use in order to investigate ideas about capacity. Amy observed what the children were doing and asked questions, made suggestions or gave them further resources to support their learning.

> Callum and Joshua filled bigger boxes with small toys and found they couldn't count that number. I don't think it mattered too much though. They were enjoying the practical experience of filling a container, they were practicing judging when a container is 'full' and they saw how they could fit more small toys in a box and less big toys.(Amy, reflective account of lesson, 2006/7)

Kate also become increasingly confident about letting children take greater ownership of their mathematics.

> I was really pleased – my upper group have finally started to work through a problem systematically on their own initiative. I didn't mention it this week as I had not really thought of that approach and they did it anyway! So we shared their systematic approach as a class. (Kate, reflective account, 2006/7)

In this instance, Kate had not intended that the children should investigate the problem in their own way. However, she was clearly happy that they did so and felt confident to discuss their strategies with the class.

Observations of the participants' teaching, and their reflective accounts showed that they became more able to respond contingently during their mathematics teaching. There was also convincing evidence from reflective accounts that the participants saw the ability to act contingently as a factor in effective teaching and that they believed they had developed in this respect over the course of the project.

> I am more experienced, so I am aware of children's common misconceptions, and can therefore adapt in response contingently, or plan for these. Generally I think there is more contingent teaching going on and I am more confident to be flexible. I can respond quickly to a child by setting up an activity I know will extend from what they are doing. (Amy, group interview,2006/7)

Kate also thought that she had become more responsive to how children reacted to her teaching.

> Quite a lot of the things that I remember talking about arose out of what the children did. One of the children who came up to write something on the board got something wrong and if he hadn't I possibly wouldn't have made that a focus. (Kate, interview, 2007/8)

By the end of her second year of teaching Jess felt that responding contingently in her mathematics teaching was something she was able to do automatically.

> I just think about contingency as a question that I hadn't thought of, that's just what you automatically do in anything, like thinking on your feet. (Jess, reflective account, 2006/7)

Conclusion

The participants' use of the contingency dimension of the KQ framework focused their thinking on the way in which they responded to unplanned-for events in their mathematics teaching. There was convincing evidence that the participants recognised the ability to act contingently during mathematics lessons as a factor in effective teaching. Over the four years of the study the participants became more able to respond helpfully to children's errors and make better 'unplanned-for' use of resources. They became more proficient at understanding, discussing and basing their teaching on children's methods and ideas. Participants' also began to adopt a more enquiry-based approach to their mathematics teaching which was more likely to require them to act contingently. These developments in participants' ability to act contingently were underpinned by development in their mathematical content knowledge. In directing participants' reflection towards their contingent actions, the KQ played a role in the development of this aspect of mathematical content knowledge.

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BSRLM Geometry working group: Establishing a professional development network to support teachers using dynamic mathematics software *GeoGebra*

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The embedding of technology into mathematics teaching is known to be a complex process. *GeoGebra*, an open-source dynamic mathematics software that incorporates geometry and algebra into a single package, is proving popular with teachers - yet solely having access to such technology can be insufficient for the successful integration of technology into teaching. This paper reports on aspects of an NCETM-funded project that involved nine experienced teachers collaborating in developing ways of providing professional development and support for other teachers across England in the use of *GeoGebra* in teaching mathematics. The participating teachers tried various approaches to better integrate the use of *GeoGebra* into the mathematics curriculum (especially in geometry) and they designed and led professional development workshops for other teachers. As a result, the project initiated a core group which has started to be a source of support and professional development for other teachers of mathematics in the use of *GeoGebra*.

Keywords: mathematics; geometry; ICT; technology; teaching; professional development; CPD; *GeoGebra*; NCETM

Introduction

Technology is becoming integral to mathematics teaching and learning, affording new forms of dynamic representation and communication (for an overview, see Heid and Blume, 2008). Yet it is also clear that the need for appropriate professional development to support teachers in designing technology-supported lessons remains paramount. Solely providing technology is insufficient for the successful integration of new dynamic tools into teaching. As Cuban, Kilpatrick and Peck (2001) report, providing access to equipment and software does not necessarily lead to widespread teacher and student use of the technology.

Yet there is evidence that appropriate professional development opportunities and collegial support can boost teachers' willingness to integrate technology into their teaching and can support their capacity to develop successful technology-assisted teaching practices (Heid and Blume, 2008). Part of this might entail aiding teachers in understanding the affordances, constraints, and general pedagogical nature of such new representational resources in relation to the specific topics in school mathematics (Hohenwarter and Jones, 2007; Ruthven and Hennessy, 2002).

This paper reports on selected components of a project funded by the *National Centre for Excellence in the Teaching of Mathematics* (NCETM) for England. The project involved nine experienced teachers collaborating in developing ways of providing professional development and support for other teachers across England in the use of the open-source dynamic mathematics software *GeoGebra* in teaching mathematics. In what follows, a brief overview is given of the software *GeoGebra*. The bulk of the paper documents selected elements from the forms of professional development and support for other teachers that were developed as part of the project

An overview of GeoGebra

GeoGebra (Hohenwarter, 2002; Hohenwarter and Preiner, 2007) is a free-to-use open-source dynamic mathematics software that incorporates geometry and algebra into a single package by providing an integrated connection between the symbolic manipulation and visualisation capabilities of CAS (Computer Algebra Systems) and the dynamic changeability of DGS (Dynamic Geometry Systems). It does this by providing not only the functionality of DGS (in which the user can work with points, vectors, segments, lines, and conic sections) but also of CAS (in that equations and coordinates can be entered directly and functions can be defined algebraically and then changed dynamically). These two capabilities are characteristic of *GeoGebra* which, as shown in Figure 1, provides two windows in which each object in the algebra window corresponds to an object in the geometry window, and vice versa (for more on this, see Hohenwarter and Jones, 2007).

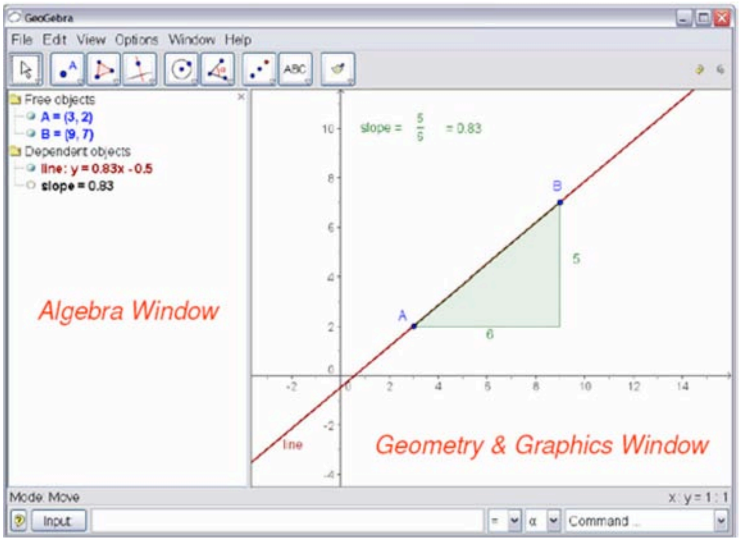

Figure 1: *GeoGebra* screen showing both the algebra and geometry windows

The upcoming update to *GeoGebra* is, at the time of writing, nearing release (see Kreis, 2009). New features include the addition of spreadsheet capabilities (linked to existing capabilities in algebra and geometry), plus, amongst other things, statistics functions and support for the use of complex numbers. In addition, custom animation of objects is becoming fully integrated into *Geogebra* with the provision of animating sliders with which the user can specify the increment, the speed, and what happens when a boundary is reached (at which point the object can bounce, or repeat). More information on *GeoGebra*, including updates and sources of teaching ideas, can be found at www.geogebra.org

The project to establish a professional development network in England

The project team for this NCETM-funded project was made up of researchers and teachers. The principal aims were to nurture in England a professional development network around the use of *GeoGebra*, to find ways in which the use of *GeoGebra* can be better integrated into the mathematics curriculum, to develop CDP workshops, and to enhance the professional opportunities for participating teachers by supporting them in giving workshops for other teachers and involving them in original research and in conferences and other forms of research dissemination.

The project was informed by several theoretical ideas, primarily the notion of communities of inquiry (Jaworski, 2006). The methodological framework is that of the design experiment (Gravemeijer, 1994). Data are from interviews with participants and the analysis of video recordings of the CPD workshops.

The facets of the project reported below are those concerned, first, with efforts made by the participating teachers to better integrate the use of *GeoGebra* into the mathematics curriculum and, second, concerning how the project team designed professional development workshops. More details are in Hohenwarter and Lavicza (2007) and the full project report presented in Lavicza, Hohenwarter and Lu (2009).

Integrating the use of *GeoGebra* into the mathematics curriculum

Given that much current use of dynamic geometry software is in upper secondary school mathematics, this component of the project examined the geometry requirements of mathematics curriculum in England to find ways in which *GeoGebra* might be used with younger pupils in the primary and lower secondary school years. The initial stages of the project showed that *GeoGebra* offered the opportunity for teachers of the youngest pupils to work with, and extend their knowledge of, basic 2D shapes by using pre-prepared files (or their own) with the interactive whiteboard or with an adult working with a small group on a computer. Later on, pupils were shown how to develop ideas themselves from base files or how to create their own examples. Work on the project suggests that it is possible to use *GeoGebra* in teaching many of the concepts found in the geometry area of the school curriculum, offering benefits to pupils such as developing a good vocabulary, being able to experiment with ideas more rapidly than drawing by hand, produce accurate drawings, and gaining instant feedback. An example developed during the project of a classroom task involving reflection in a line is shown in Figure 2.

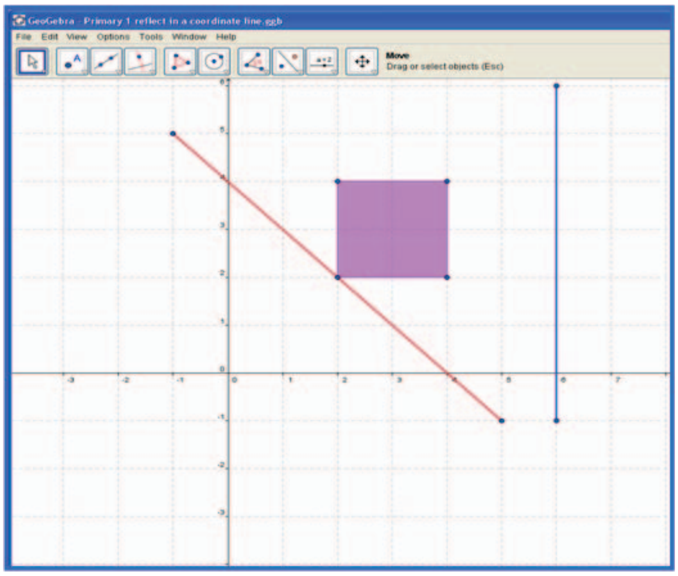

Figure 2: *GeoGebra* screen showing a reflection task

More examples of tasks that were developed during the project can be found through the online *GeoGebra Wiki* at: www.geogebra.org/en/wiki

Developing and providing professional development for teachers

In this component of the project, participating teachers not only contributed to the development of professional development materials, they also led workshops for other teachers. The workshops developed during the project included providing an introduction to *GeoGebra*, with hands-on activities during which some basic problems from geometry and algebra were tackled by the participants.

A particularly promising approach to stimulating professional conversations about teaching approaches is the pedagogical framework, illustrated in Figure 3, which was developed during the project. This pedagogical framework was presented at the CPD workshops and provided both a way of structuring discussion and a prompt for further discussion and further work.

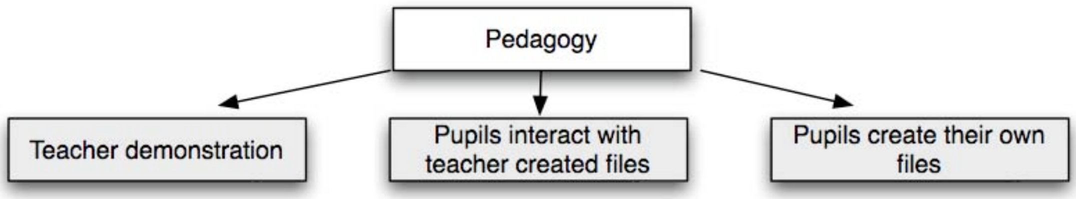

Figure 3: Pedagogical framework of approaches with *GeoGebra*

Examples of the use of each of the three approaches summarised in Figure 3 were presented at the range of CPD workshops conducted as part of the project. In the first of these approaches, that of teacher-demonstration, the teacher engages students in discussing a dynamic construction using *GeoGebra*. With this approach, the teacher can ask questions about the objects on the screen and get students to explain what they might expect would happen if some parts of the configuration were moved or changed. Then either the teacher or some students can change the construction to check such predictions.

Such demonstrations, it was found during this project, allow teachers who have little experience with using technology in the classroom to experiment with the technology with relatively small risk. In addition, this kind of use of technology requires less change in the classroom setting and needs fewer resources than organising classes of students into a computer room or when using a class set of laptops in the regular classroom.

When teachers become more comfortable with computers the second approach captured in Figure 3 entails teachers providing previously created *GeoGebra* files for their students. With such teacher-created files, students can experiment with dynamic objects. Such an approach provides clear boundaries for students and student time is not spent setting up the task. Instead, students can spend time exploring the mathematics that is central to the task. There is quite some teacher control over the material, but the approach also brings in opportunities for creative thinking and problem solving by students.

It was recognised in the CPD workshops that this approach of students using teacher-created files to explore problems may well mean teachers transforming the way they are teaching. It entails teachers experimenting with the content of their lessons and adopting a more investigative approach. It might also mean highlight different aspects of mathematics and might entail working with different starting point, both of which should stimulate discussion and the sharing of ideas in the classroom.

Yet, while there is the potential for student engagement with such teacher-created files, it may be that some such files may engender little more the procedural thinking in students. There is also the danger of a lack of student engagement, with no more than random play (perhaps unproductively) with the file. What is more, students may not relate to the problem as there is a lack of ownership which could be restrictive.

The third approach captured in Figure 3 entails students creating their own files, perhaps for other students to tackle. This approach provides ownership of the work and engages a different sense of problem solving and thinking by creating that ownership. There is also the development of independence – in learning how to use *GeoGebra*, and with more scope for student creativity and discovery. Students are being imaginative, creating their own 'What ifs?', and, as such, may be more likely to go and use *GeoGebra* for themselves, perhaps at home. No doubt there are risks too with such an approach; something that discussed by participants during the CPD workshops. For example, students creating their own files may well be time consuming, and such creations may not have the desired impact.

Outcomes and discussion

According to a recent research report on the state of overall continuing professional development of teachers in England (Pedder, Storey and Opfer, 2009: 13), "teachers place most value on CPD that involves experimenting with classroom practices, working collaboratively, and adapting approaches in the light of pupil/peer feedback and self-evaluation". In a similar vein, the recent NCETM project on effective CPD in Mathematics Education (NCETM, 2009: 3) reports that "teachers valued practical advice that was directly applicable to the classroom, including resources and banks of resources that they could use with minimal adaptation. In many cases they valued having attention drawn to the use of practical equipment and ICT resources which support mathematical thinking and reasoning". This project sought to all do these things, and more.

The project examined various ways of supporting teachers in building their capacity to develop successful technology-assisted teaching practices. One component of the project entailed working on ways in which *GeoGebra* could be integrated into the mathematics curriculum in England and one outcome has been the developing and collecting of classroom materials that can be used in mathematics teaching.

Another component of the project involved the participating teachers not only contributing to the development of CPD materials, but also leading workshops for other teachers. With a group of nine enthusiastic participating teachers, the project has initiated a core group that is ready to continue developing support and CPD for other mathematics teachers in England.

Concluding comment

It is fitting to conclude with noting that the group of teachers who collaborated in the project have become interested in research and in sharing their experience with other teachers at conferences and through various publications. We hope that reports on this project further contribute to nurturing a community of teachers and researchers in England who are interested in developing and using open-source technology in schools and in teacher education.

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BSRLM geometry working group

The BSRLM geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. Suggestions of topics for discussion are always welcome. The group is open to all. See: www.bsrlm.org.uk/workinggroups.html