ISSN 1463-6840

Proceedings of the British Society for Research into Learning Mathematics

Volume 26 Number 2

Proceedings of the Day Conference held at the University of Bristol, Saturday 17th June 2006

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Published and distributed by the British Society for Research into Learning Mathematics.

20 Bedford Way, London, WC1H 0AL, England.

Editor: Dave Hewitt, School of Education, University of Birmingham, B15 2TT

ISSN 1463-6840

Proceedings of the British Society for Research into Learning Mathematics

Volume 26 Number 2. June 2006

Proceedings of the Day Conference held at the University of Bristol
Saturday 17th June 2006

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PROBING UNDERSTANDING THROUGH EXAMPLE CONSTRUCTION: THE CASE OF INTEGRATION

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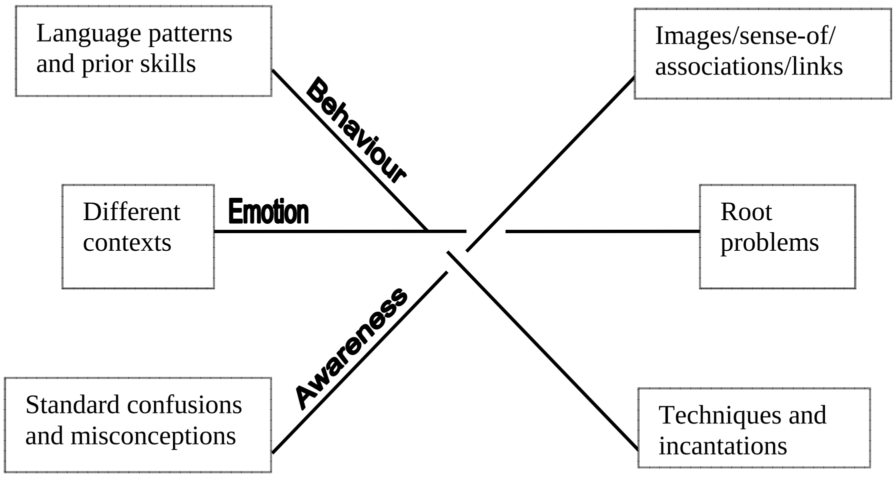
Activities that reveal something about learners as they become aware of aspects of mathematical concepts that were not previously focussed upon can be very useful for the learners themselves as well as for teachers and researchers. In this paper I consider example construction tasks as research probes to reveal learners' awareness of the concept of integration. Forty students studying A-level, engineering, mathematics and education have been invited to construct relevant mathematical objects meeting specified constraints. What learners choose to change in mathematical examples reveals the dynamics and depth of their awareness and acts to promote and enrich their appreciation of the concept.

INTRODUCTION

Coming to understand mathematical ideas can take different forms for different learners. However, a common feature of understanding seems to be that it is highly situated, in the sense that learners may be well-equipped to work on standard textbook problems that appear in familiar contexts, and yet become incapacitated when faced with novel situations. Learners do not seem to be able to cope with situations beyond the familiar. Knowing to act in the moment (Mason & Spence, 1999) requires awareness that brings knowledge relevant to the situation to be acted upon. Thus, I see conceptual understanding as evidenced by behaviour which not only uses knowledge conceived in a familiar setting to solve routine problems correctly, but more importantly, by behaviour which extends that knowledge appropriately and efficiently into unfamiliar situations. Dealing effectively with novel situations depends on which aspect(s) of the concept/idea become(s) the focus of learners' attention, thus regarded as important. In this paper, I explore learners' awareness and understanding of integration by having them construct relevant mathematical examples meeting specified constraints.

Knowledge and Understanding

Understanding of a mathematical topic involves learners getting a sense of it in relation to their past experience. Tall & Vinner (1981) use the term *concept image* to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." Based on Gattegno's (1987) assertion that awareness is what is educable and enriching the notion of personal *concept image* with the three interwoven dimensions of human psyche, a framework referred to as 'structure of a topic' (Mason & Johnston-Wilder, 2004) was developed to describe how a mathematical topic is conceived. It is used to analyze learners' understanding of mathematical topics, as well as to prepare one to teach a topic.



'Structure of a topic' framework (Mason & Johnston-Wilder, 2004)

The framework encompasses three strands: behaviour, emotion and awareness which are closely associated with the more familiar terms *enaction*, *affect* and *cognition*. Behaviour is trained through practice but training alone renders the individual inflexible. Flexibility arises from awareness which informs and directs behaviour. Learning then involves educating awareness which in turn directs appropriate behaviour. Energy and motivation to learn arise from the harnessing of individuals' emotions. Marton (Marton & Booth, 1997) regards learning as making distinctions, both discerning something from, and relating it to, a context. A particular form of active awareness is discernment of variation. This fits with a view of mathematics as being essentially about the study of *invariance in the midst of change*. In the context of learning mathematics, Watson & Mason (2005) suggest that learning involves extending awareness of *dimensions of possible variation* associated with tasks, techniques, concepts and contexts, and extending awareness of the *range of permissible change* within each of those dimensions.

Example construction and awareness of dimensions-of-possible-variation

An integral part of effective mathematics instruction is the use of examples to illustrate and clarify mathematical concepts. While teachers may be attending to the generic aspects of examples, learners may develop restricted thinking that only those particular examples are appropriate, unless explicit attempts were not taken to draw learners' attention specifically to what is learned. Getting learners to construct examples can reveal aspects which dominate learner attention and thus, what they regard as important. By prompting learners to construct mathematical examples, what they choose to change reveals dimensions, depth and scope of their awareness. Although learners do not often encounter this type of question in their normal learning environment, I conjecture that the task itself could offer opportunities to experience structure of mathematical examples, to discern what is invariant and what can be varied and more importantly, and so to reveal their awareness of mathematical concept.

METHOD

Forty students were interviewed in pairs with regard to their understanding of integration, eighteen of which were first-year undergraduates studying mathematics, engineering or PGCE mathematics in three universities in South Midlands and the remaining 22 were studying A-level in a school in the same area. The aim was to reveal the range of responses regarding the understanding of integration in different learners. The semi-structured interviews were tape-recorded and transcribed verbatim. The students were also invited to construct relevant mathematical objects meeting specified constraints with the aim of revealing their awareness of the topic. Learners' responses were analyzed to find out aspects of the topic that were the focus of their attention and how shifting their attention to what must remain invariant and what can vary reveals the structure of their awareness.

DATA ANALYSIS

Analysis of responses from Jim and Dinah (pseudonyms), first year PGCE Mathematics students, to questions on understanding of integration suggests that their responses were highly dominated by behavioural aspect of integration, namely techniques. They displayed fluency in language patterns and techniques of integration.

Researcher: What does the word "integration" mean to you?

Jim: Reverse of differentiation, summation, working out areas, volumes, moments of inertia.

Researcher: Do you have any images/conversations when you see the sign?

Jim: Apart from use in area under a curve and the first principle, but then you quickly get on to the rules and different techniques.

Dinah: That is an integration question, you have to apply particular rules to find [the] value.

Both Jim and Dinah displayed awareness of things to watch out for when doing integration questions, in which they stressed technical aspects of integration.

Researcher: What sorts of things have you discovered you need to watch out for when you are doing integration?

Jim: You've got to work out what rule/algorithm to apply to a particular question. For example integration by substitution, the formula for substitution, the kind of complication you don't get in differentiation.

Dinah: Need to look for limits because your answer differs when you got limits.

The following task was then given to find out about the nature of their awareness and the focus of their attention when working on a problem:

Given that $\int_0^2 (1-x)dx = 0$. Can you find another example like this where the answer is 0?

Jim constructed examples by changing the upper limit and the function while Dinah changed just the function.

Jim: $\int_0^3 3 - x^2 dx$. Dinah: $\int_0^2 (x-1)dx$.

Jim's attention seems to be focussed on the upper limit, increasing it by 1 and changing both terms in the function. He seems to be focusing on the relationship between the limits and the terms in the function. Dinah constructed her example by simply reversing the order of the terms in the function and maintaining the limits, which suggests a limited awareness of or confidence about what can vary. When asked to give another example, both worked on Jim's example, changing the upper limit and the function.

Researcher: Can you give another example?

Jim: I guess $\int_0^4 (4 - x^3)dx$. I haven't worked out the general rule.

Dinah: No, it doesn't work. [Checks]

Jim: It would have to be … 16. Make that *any* number you like. So 16*x* … [calculates]

Dinah: No, it should be $x - 4x$. Or you can make it $\int_0^4 (x - 4x^3)dx$. It won't work.

Jim: $16 - x^3$, that's alright. [Checks] $16 - \frac{x^4}{4}$. Yeah … it will work if you do multiples of 4. So then we'll have $25 - x^2$ … If you do $\int_0^5 (25 - x^4)dx$ it works. [Checks] 25 - $\frac{5^4}{4}$. No, it has to be another multiple of 5 … 125.

Researcher: Can you give another example?

Jim: $\int_0^6 (6^4 - x^5)dx$. Dinah: [Checks] Yeah.

When probed to give another example, Jim increased the upper limit and changed the terms in the function. He discerned the limits as a variable dimension. He seemed to be *guessing* at a pattern between the upper limit, the first term and the second term, admitting that he has not worked out the general rule for the pattern. Having been corrected by his partner, Jim then *adjusted* the first term to accommodate the changed upper limit. Based on his first two examples, Jim *looked for pattern* and generalized (partially correctly) for the third and subsequently, *spotted the pattern* and *generalized*. Dinah checked her generalization by constructing examples consisting of functions $y = x$ and $y = - x$ from -1 to 1 and suggested that a general example could have one positive and one negative limit.

Researcher: Can you give me a general example for which the answer is zero?

Dinah: $\int_{-1}^{1} xdx$ [Checks]. $\int_{-1}^{1} -xdx$, let me find out …is that right?

Researcher: What is the most the general example you can think of for which the answer is zero?

Dinah: If you make one negative and one positive it might work.

Researcher: What can you change?

Jim: The limits and the expression.

Researcher: What happens when you change the limits or the expression?

Dinah: When you change [upper] limits, take multiples of limits. For example 5 [you take] 125.

Jim: So that would be $\int_{0}^{n} (n^{n-2} - x^{n-1})dx$. You could prove that by induction.

Although both Jim and Dinah displayed awareness of some dimensions that can vary in the example, the fact that other dimensions like variable *x* and method of integration were not mentioned suggested that they did not mark those dimensions as variable and were less likely to appreciate the interconnectedness of these dimensions. This reveals something about the nature of their awareness, which seems to be limited and fragmented.

Although area was mentioned when asked what integration meant, Jim did not display awareness of this association when dealing with integration question. Even though he might be aware of this useful connection, the fact that it was not mentioned when constructing examples like the one given suggests it did not come to mind in the moment. Extra triggers were required to invoke awareness of the integral as area, which called for shifts in attention to seeing connections and relationships.

Researcher: Why is it coming to zero?

Jim: Because this is different area under the curve. [Long pause] We have actually integrated across … We probably got it wrong, have we? $y = 1 - x$ is that line [sketches] So in fact we've got two areas that have been taken away from each other. So in fact the area under the curve… so should we not have integrated from 0 to 1?

Researcher: Now can you think of another example?

Jim: Any value between there [points to limits of $y = 1 - x$]. So where you got the area is opposite sides of the *x*-axis. In this case you should integrate it from there to there [points to limits].

DISCUSSION AND CONCLUSION

The different ways in which learners construct mathematical objects can reveal something of the nature and structure of differences in how learners experience and understand what they are learn. Particularly, learners' awareness is revealed through aspects in a problem on which they focus their attention and thus, presumably, regard as important in that moment at least. Getting them to talk about aspects which dominate their attention can reveal dynamics of their thinking. By becoming aware of features not previously at the focus of their attention, learners who appeared to be expecting to be 'tested' about their knowledge of integration in fact revealed to themselves aspects of integration that were not previously salient to them (according to their comments at the end of the interview). Learners can have fragmented awareness of concepts which can differ from what they say and what they do.

The use and exploitation of the 'structure of a topic' framework suggest that there are aspects of a topic that become the focus of learners' awareness and thus central to their attention when doing mathematics. These aspects that are stressed more than others may prevent them from becoming simultaneously aware of other aspects, which would help them gain a richer understanding and appreciation of the topic. The extent to which learners' awareness/attention is drawn explicitly to what is being learned can act to promote and enrich their understanding.

Prompting learners to construct examples not only reveals focal aspects of the object but also their awareness of dimensions of possible variation in mathematical examples so that learners can experience the structure of what is exemplified. Sensitivity of probes is important in revealing more or different things learners could be aware of because different probes get different answers from learners. These probes account for learning about the learners from the probes, as opposed to learning about the probes from learners' responses in other kinds of tasks.

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USING DISCURSIVE PSYCHOLOGY IN RESEARCH IN MATHEMATICS CLASSROOMS: WHAT CAN BE SEEN AND WHAT IS OBSCURED?

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Discursive psychology has emerged as an anti-cognitivist, anti-realist, anti-structuralist perspective on cognition. This approach includes both a theorisation of the role of discursive practice in thinking, and a methodological approach to the investigation of psychological questions. Discursive psychology has informed much of my research into the role of multilingualism in the teaching and learning of mathematics. In this paper, I reflect on what the adoption of this perspective has allowed me to see, as well as what it may have obscured.

INTRODUCTION

There has been a long-standing interest in the role of language and discourse in mathematics classrooms, from considerations of the nature of mathematical language, to the role of talk and interaction in teaching and learning. Despite this interest, mathematics educators have been slow to work with the various established approaches to the theory and analysis of discourse found in applied linguistics or sociology (with the exception of work based on systemic linguistics). My own research is based on one such approach: discursive psychology (Edwards and Potter, 1992; Edwards, 1997). In this paper, I will consider what discursive psychology has enabled me to see, as well as what it has obscured, with the aim of highlighting its potential usefulness for research in mathematics education.

DISCURSIVE PSYCHOLOGY

Discursive psychology, which develops ideas in ethnomethodology and conversation analysis, has been described as offering an anti-cognitivist, anti-realist, anti-structuralist account of the relationship between discourse and psychological process, such as thinking, meaning or remembering [1]. In the context of mathematics classrooms, these points might have the following implications.

- *Anti-cognitivist*: entails a shift from a focus on 'what happens in the mind' (as an individual mental process) to how 'what happens in the mind' is done through discursive practice (as a socially organised process); thus, mathematical thinking and meaning are jointly produced through interaction. Organisation of this interaction is primarily social, and only secondarily based on individual mathematical thinking.
- *Anti-realist*: reality is seen as being reflexively constituted through interaction. Thus mathematics is not pre-given, but is brought about through talk. Rather than mathematical meaning being pre-determined by words, symbols or diagrams, participants 'give sense' to these things through their interaction.

- *Anti-strucuturalist*: following the preceding point, mathematical meaning and the organisation of mathematical interaction are situated, emerging from preceding interaction, rather than in simplistically predictable ways.

Consider the following exchange between a pre-service teacher and a 7-year old girl, reported by Sfard (2001, p. 19):

Teacher:	What is the biggest number you can think of?
Noa:	Million.
Teacher:	What happens when we add one to million?
Noa:	Million and one.
Teacher:	Is it bigger than million?
Noa:	Yes.
Teacher:	So what is the biggest number?
Noa:	Two millions.
Teacher:	And if we add one to two millions?
Noa:	It's more than two millions.
Teacher:	So can one arrive at the biggest number?
Noa:	Yes.
Teacher:	Let's assume that *googol* is the biggest number. Can we add one to googol?
Noa:	Yes. There are numbers bigger than googol.
Teacher:	So what is the biggest number?
Noa:	There is no such number!
Teacher:	Why there is no biggest number?
Noa:	Because there is always a number which is bigger than that?

In discussing this exchange, Sfard argues:

> Clearly, for Noa, this very brief conversation becomes an opportunity for learning. The girl begins the dialogue convinced that there is a number that can be called 'the biggest' and she ends by emphatically stating the opposite: 'There is no such number!'. The question is whether this learning may be regarded as learning-with-understanding, and whether it is therefore the desirable kind of learning (p. 19).

From the perspective I have outlined above, however, this reading is in some ways problematic, largely because it does not take sufficient account of the situatedness of interaction and the joint nature of what is produced. It is hard to say, for example, that Noa is 'convinced' that there is a biggest number. Rather, the teacher's questions invite her to offer big numbers. It would be interactionally awkward for Noa to respond to the first question 'there is no such thing', since this would be to challenge the understanding and authority of a perhaps unfamiliar teacher. Furthermore, the teacher elides the categories ‘the biggest number you can think of’ and ‘the biggest number’, so that Noa’s responses can be seen as belonging to the former, rather than the latter. This elision is partly brought about by the patterned nature of the teacher’s questions. This patterning also leads to Noa’s statement ‘There is no such number!’ since by repeatedly returning to the same question ‘So what is the biggest number?’, the teacher signals a problem with Noa’s responses. Providing or saying that there is a biggest number is ‘troubled’ by this revisiting. Noa must come up with an alternative.

A discursive psychology perspective, then, rather than being concerned with what Noa knows, is convinced about, thinks or learns is interested in how discursive practice is used to construct these things. From this perspective, Noa’s ‘conviction’ that there is a biggest number is an interpretation based on her participation in a patterned series of questions. Similarly Noa’s eventual formulation of a mathematically more acceptable response cannot be seen as an isolated epiphany; it emerges from several features of the exchange, including the cycles of questions and the repeated troubling of the key question. ‘Noa’s’ thinking is jointly produced with the teacher. Whilst this position is in some way similar to Sfard’s (pp. 46-49), who seeks to understand the relationship between the discursive and the cognitive, it goes further, in seeing (joint) cognition itself as discursive.

DISCURSIVE PSYCHOLOGY IN MY OWN RESEARCH

The broad principles of discursive psychology lead to a methodological approach that involves looking at how ‘what happens in the mind’ is done in specific situations. This approach entails detailed examination of extended sequences of naturally occurring interaction – naturally occurring because cognition is seen as situated; extended sequences because preceding talk provides a big part of the context of current interaction. Analysis seeks to uncover how participants use talk to do remembering, knowing, meaning etc. One strand of work, for example, has looked at how narratives are produced in particular ways to suit particular circumstances – to undercut potential criticism, for example (see, for example, Edwards, 1997). These ideas offer the basis for an approach to data collection and analysis, although in any particular piece of research, the approach must be developed to suit the specific needs of the project.

I have adopted a discursive psychology perspective in response to issues arising from my research in multilingual, multicultural classrooms. In many cases, the participants with whom I was working were still developing proficiency in English and were from cultural backgrounds with which I was not familiar. This situation makes it difficult to simply ‘read off’ what students mean from the words they use. By shifting to a focus on how mathematical thinking or learning is jointly produced through interaction, rather than a focus on *what* students were thinking, interpretation becomes less problematic (although still interpretation).

In my research so far, I have looked, for example, at primary school students' interaction as they work on a task of jointly writing arithmetic word problems (Barwell, 2003, 2005). My analyses have, amongst other things, explored how students draw on discursive practices relating to genre, narrative, formal mathematics and written English. I have analysed in some detail, for example, how narrative accounts are used to give sense to the rather stilted situations depicted in word problems. In the following exchange, two Year 5 monolingual students, Vicky and Emily, are writing a problem about supermarkets (for transcription conventions, see [2]):

V	um/ two/ chil-/
E	-dren
V	go to the supermarket
E	huh?/ I'm going to a market on sunday to get some new shoes and some steak
V	where?
E	Eastern market/ to get some new shoes and/ no/ I mean/ [to get some steak
V	[two children go/ to/ the
E	and I'm going to the market on sunday to get some steak and then I go to **town** to get some new shoes/
V	twen-/ twenty five pound
E	gosh/ you must be rich/ fifty p. (*laughs*)
V	my mum spends/ three hundred and seven pounds at Tesco's
E	oh my gosh/ my mum spent/ my mum spent four hundred in Tesco's before/
V	but the thing is my mum keeps doin' it
E	I **know**/ they goes to the/ supermarket/ like **every day**/ to get **munchies**/ sweets/ (*tuts*)
V	anyway/ two children go to the supermarket with twenty five pounds
E	yeah
E	yeah
V	and
E	fifty p. (*laughs*)
V	buy/ washing up liquid
E	(*laughs*) costing/ hang on

V costing (discussion continues…)

This extract depicts an interleaved discussion in which the two students (a) begin to put together a word problem about two children going to the supermarket with twenty-five pounds to spend (b) construct narrative accounts about supermarkets involving their mums. These two strands form an interesting contrast. The stories about their mums are authoritative. Authority is constructed, for example, with relevant narrative detail (e.g. how much is spent; *why* their mums go). Their jointly produced accounts are examples of a common pattern in relaxed social talk, that of 'topping' stories: each participant outdoes the preceding contribution. The effect is convergent, however, with Emily eventually talking about both the mums together. These features contrast in some ways with the construction of the word problem, which is more hesitant, less authoritative, and without the same degree of human detail. Later in the discussion, however, Vicky says, referring to the two nameless children in their word problem: "**they're** going shopping for their mum", forming an explicit link between the students' narrative accounts and their work on the mathematical task. Thus the narrative accounts are recruited as relevant in giving sense to their own word problem.

REFLECTIONS: WHAT DO I SEE AND WHAT HAS BEEN OBSCURED

Working with the ideas of discursive psychology has allowed me to see how children in culturally and linguistically diverse classrooms do mathematics. These ideas particularly make visible the intricacies of students' discursive practices in making sense of mathematics and highlights the ways in which these practices are used to do mathematical thinking. In the case of Noa, for example, the pattern of questioning can be seen as structuring the mathematical thinking of the two participants. In the case of Vicky and Emily, their narrative accounts are used to give sense to a word problem, as well as to identify with each other. Any perspective, of course, obscures as much as it illuminates. One possible aspect of interaction that is to some extent obscured by discursive psychology is the macro structures of class, ethnicity or institutional orders. In another word problem discussion, for example, which is in many ways similar to the extract I have used in this paper, the participants are a White student and a Hong-Kong Chinese student who is new to the UK and a learner of English (Barwell, 2005). A series of narrative accounts emerge concerning how much pocket money they receive. The White student talks about earning increasingly large amounts of money. Instead of a pattern of 'topping' stories emerging, however, the Chinese student talks about how her father is sometimes unemployed and how she has to work to earn any pocket money. Their accounts are in tension, though still used to give sense to their word problem. It is possible that the difference between these two instances of the use of narrative accounts may be due to the different ethnic composition of the two groups. Discursive psychology makes it hard to explore this possibility. Nevertheless, discursive psychology offers a rigorous theoretical and methodological approach with which to explore how discursive practice is implicated in mathematical thinking and learning.

NOTES

1. The three antis are derived from a response given by Margaret Wetherell as part of a UK Linguistic Ethnography Forum colloquium at the annual meeting of the British Association for Applied Linguistics, Bristol, 15-17 September 2005.

2. Transcription conventions: Italics indicates emphasis. / and (.) indicate short pauses. ? is for question intonation. [for concurrent speech. :: shows extended sounds. (hh) indicates laughter.

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Hewitt, D. (2006) Proceedings of the British Society for Research into Learning Mathematics 26(2) June 2006

PEDAGOGICAL SENSITIVITY AND PROCEDURAL THINKING: AN UNEASY RELATIONSHIP?

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We examined the relationship between subject-matter knowledge and pedagogical content knowledge of 53 in-service mathematics teachers in the context of their written responses to a question that involved: solving the equation $|x|+|x-1|=0$, *examining a flawed student solution and providing feedback to the student. Here we focus on a group of scripts characterised by pedagogical sensitivity but constrained mathematically (substantively and meta-cognitively). Through examination of examples from the data we demonstrate, and discuss implications of, some of these constraints: insistence on standard procedural methods, inappropriate contextualisation of otherwise commendable pedagogical practices and inadequate reflection on student thinking.*

Work on how the preparation of teachers can facilitate the development and employment of their subject-matter knowledge and beliefs and knowledge about pedagogy gained significant theoretical momentum from the 1980s onwards – particularly with Shulman's (1986, 1987) now seminal seven-type taxonomy of teacher knowledge. Crucially this taxonomy included *pedagogical content knowledge*, 'the ways of representing the subject that make it comprehensible to others' (1986, ibid, p9). The 'most useful forms of representation' of mathematical ideas as well as an understanding of students' background, previous knowledge, preconceptions and potential difficulties were included in Shulman's definition. Latter-day explorations of the manifold nature of teacher knowledge highlighted *sensitivity to students* and *mathematical challenge* (Jaworski 1994) as aspects of teachers' practice where their pedagogical content knowledge is most evident. Furthermore this and other studies – for example, see Thompson (1992) for a review – acknowledge that, given the overt discrepancy between theoretically and out-of context expressed teacher beliefs about mathematics and pedagogy (e.g. in interview-based studies) and actual practice, *teacher knowledge is better explored in situation-specific contexts*.

Our work explores teacher knowledge in such a context. In particular the study we report here[1] explores how pedagogical practice is influenced by ways of knowing about mathematics (e.g. in the light of Mason and Spence's (1999) *knowing-to act mathematically*. The particular examples from the data we employ in this paper are also seen in the light of the distinction between *procedural and conceptual mathematical thinking* (Hiebert, 1986).

[1] An ongoing small-scale study conducted by a team of researchers based in Greece and the UK and supported by an ERASMUS grant and by the University of Athens (ELKE). The data presented here have been translated from Greek. The team consists of three mathematicians with complementary backgrounds: a university lecturer and researcher in mathematics education; a mathematics teacher currently completing a doctorate in mathematics education; and, a university lecturer and researcher in mathematics with a currently developing interest in mathematics education research.

THE STUDY

Context and Data Collection. The data we draw on in this paper are the written responses to one (Fig. 1) out of nine questions set in an exam taken by 53 in-service secondary mathematics teachers for the purpose of selection for a Masters in Mathematics Education programme. All participants are mathematics graduates with teaching experience that ranges from a few to many years and most have attended in-service training of about 80 hours.

In a mathematics test students were given the problem:

"Solve the equation: $|x| + |x-1| = 0$"

a. What do you think the examiner intended by setting this problem?
b. A student responded as follows:

"It is true that

$$|x| + |x-1| = 0 \Leftrightarrow \left(|x| + |x-1|\right)^2 = 0 \Leftrightarrow x^2 + \left(x-1\right)^2 + 2|x(x-1)| = 0 \Leftrightarrow$$

$$x^2 + x^2 - 2x + 1 + 2|x(x-1)| = 0$$

Case 1: $x(x-1) \leq 0$

Then

$2x^2 - 2x + 1 - 2x^2 + 2x = 0 \Leftrightarrow 1 = 0$ Impossible.

Case 2: $x(x-1) \geq 0$

Then

$$2x^2 - 2x + 1 + 2x^2 - 2x = 0 \Leftrightarrow 4x^2 - 4x + 1 = 0 \Leftrightarrow \left(2x-1\right)^2 = 0 \Leftrightarrow x = \frac{1}{2}$$

Therefore the solution of the equation is $x = \frac{1}{2}$."

What comments would you make to this student with regard to this response?

Fig. 1

In the above the student has successfully carried out a case by case study of the equation but has failed to observe that in the second case ½ is not a solution (it does not satisfy $x(x-1) \geq 0$) and therefore the equation has no solutions. Furthermore the student has not observed that, instead of this acceptable but long-winded procedural approach (which, along with the approach that distinguishes cases according to the sign of x and x-1, we will call the '*routine*' methods), the above conclusion can be reached by simply noticing that, given the non-negative nature of absolute value (| |), $|x|$ and $|x-1|$ must be simultaneously zero if they add to zero – which is impossible.

We call this latter approach the '*optimum*' solution as it is shorter and draws on a deeper conceptual understanding of | |: selecting a crucial attribute of the entity, its non-negativity, and, through mathematical logic, applying it towards the solution of the equation. Also, from a didactical point of view, acknowledging this solution offers an opportunity to discuss meta-cognitive issues such as benefiting from awareness of multiple ways of approaching a problem. Of course the procedural approach has advantages as well: it is generalisable and demonstrates substantial ability in algebraic manipulation. Furthermore teachers' preference for this approach may be grounded on the idea that at this stage of their students' learning it is too confusing to bombard them with alternative solutions when they are still struggling with the standard methods[2].

The objectives for including this question in the exam[3] were:

- To assess the candidates' subject-matter knowledge.
- To assess the candidates' sensitivity to student difficulty and needs; and,
- To assess the candidates' ability to provide adequate (pedagogically sensitive and mathematically precise) feedback to the student.

With regard the second and third of the above in part (b) of question the candidates were expected to explain how they would work with the student towards: casting doubt on whether ½ is a solution to the equation; identifying where the student's case by case exploration was flawed; reconstructing the student solution; observing that the student may have chosen an unnecessarily long-winded approach whereas other approaches may exist; suggesting the 'optimum' solution; and, juxtaposing and discussing the merits / weaknesses of each of the two solutions and the value in circumventing routine methods in favour of concise and elegant ones.

The candidates' responses were expected to operate at least at three levels: the substantive (algebraic and logical manipulations in solving equations; conceptual understanding of | |), the meta-cognitive (acknowledgement of the multiple ways in which an equation can be solved; optimal choice of solution) and the didactical (utilise the opportunity offered by the problem to discuss problem solving skills such as the above mentioned elements of meta-cognitive awareness). We perceived a candidate's preference and emphasis on the 'optimum' solution to be evidence of mathematical subtlety combined with sensitivity towards the learning situation in question.

The strength and idiosyncrasy of this type of data lies in its highly focused, situation-specific character: candidates are asked to respond to mathematically / pedagogically specific situations that are likely to occur in the mathematics classrooms they are usually operating in. We believe that this specificity (Dawson, 1999) – in contrast to posing questions pitched at a theoretical, decontextualised level – can generate authentic access to candidates' views and intended practices. Of course we also recognise that the teachers were *not* in the classroom and they had some time to think about their reaction. However we consider that the latter may allow teacher responses to be more representative of their *intentions*. We would also like to note a methodological limitation of the data: the candidates' written responses is the one and only source of evidence we have, thus far, regarding their views and intended practices.

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[2] We note that in the analysis that follows graph-based solutions have not been considered, mostly because there was no reference to this type of solution in the 53 scripts. Exploring the reasons for this absence are outside the scope of this paper.

[3] There was one set with similar intentions in the context of Calculus but in this paper we will limit ourselves to an exploration of the Algebra question only.

Data Analysis and Findings. Regarding the analysis of the 53 scripts, for each we produced an Analytical Summary, a narrative of approximately 100-200 words, in which we summarised the script's contents and evaluated the candidate's responses in accordance with the above objectives. Further scrutiny of the scripts and the Analytical Summaries led to a three-dimensional taxonomy and all scripts were characterised in terms of their perceived *mathematical, didactical and pedagogical limitations* (five, five and nine categories respectively). For the purposes of this paper we need to focus only on the first two[4]. The numbers in the brackets refer to the number of scripts allocated in each category.

MATHEMATICAL LIMITATIONS

1. Regards flaw of the student response to be the squaring in the first line (13).
2. Regards that in Case 1 $x(x-1)\leq 0$ needs to be solved – this is unnecessary (7).
3. Does not see the 'optimum' solution (6).
4. Makes technical mathematical mistakes – e.g. in algebraic manipulations (8).
5. Does not identify the flaws in the student's response (1).

DIDACTICAL LIMITATIONS

1. Does not reconstruct the student response (3).
2. In reconstructing the student's solution, in particular in order to reject ½ as an acceptable solution, proposes the use of standard procedural methods that are unnecessarily convoluted – e.g. solving inequalities, graphing the intervals in which *x* needs to belong etc. – instead of simply substituting x with ½ in the inequality and seeing it needs to be rejected. (21).
3. Presents a routine method other than the student's – see below Fig.1 (2).
4. Despite identifying the 'optimum' solution in part (a), does not refer to it in part (b) – either at all (14) or faintly (3).
5. Describes an overly general and theoretical pedagogical approach (7): offers substantively and meta-cognitively rich propositions but does not embed them in the specific situation set in the question (5 of the seven); alludes to constructivist ideas such as encouraging the student's own reconstruction of the solution but in fact simply letting the student unguided and possibly lost (1); offers limited feedback based on superficial generalisations on student's ability (1).
6. Appears to aim at the use of commendable pedagogical practices, such as exemplifying, but employs them unsuccessfully – e.g. proposes examples that are incorrect, miss the point or are potentially misleading (3).
7. Uses mathematical (terms, symbolism) or ordinary language problematically (8).
8. Focuses excessively on insubstantial, trivial aspects of the question (2).
9. Uses mathematical formalism in an over-the-top and potentially misleading way (1).

[4] For information, under Pedagogical Limitations (P categories) we listed: presenting solution without encouraging participation / discussion (21); no attempt to 'psychoanalyse' the student response (10); lacking meta-cognitive reflection (21); mere identification and correction of the mathematical flaw of the response (13); and, drawing hasty, largely unfounded and clichéd generalisations about the student's ability (3). We would like to stress that we hesitate to draw more definitive inferences from the contents of the P categories for the following reason: most P categories highlight the *absence* of a certain reference in the candidates' response –for example, absence of a reference to encouraging student participation. We feel that we cannot infer a candidate's conscious choice against student participation from this non-reference in the script. To draw such an inference we would need further and more solid evidence, e.g. from observing their classroom practice or interviewing. On the other hand we feel more confident towards acknowledging commendable pedagogical intent to scripts where there are overt references to, for example, encouraging student participation.

Here we wish to focus on the choices made by the candidates in about a fifth of the scripts that we characterised as demonstrating pedagogical sensitivity but were constrained mathematically (at the substantive or meta-cognitive level). In particular we wish to examine how these constraints may divert the candidates away from materialising their good pedagogical intentions. Our concern is not only with their substantive constraints (such as M1, M2, M4 and M5). We are mainly concerned with how insistence on the routine methods (D2 and D3) may have diverted the candidates from thinking about (M3) and/or suggesting to the student (D4) the 'optimum' solution – see exam-setters' expectations earlier.

We identified three types of the above mentioned constraints in this section of the scripts: *insistence on standard procedural methods*, *inappropriate contextualisation of otherwise commendable pedagogical practices* and *inadequate reflection on student thinking.* Five of the scripts exemplify these constraints more overtly: the scripts of candidates [23], [43], [25], [45] and [50]. Due to limitations of space we present one of these, the script of candidate [25], and summarise the rest. Candidate [25] expressed a preference for commendable general pedagogical practices but in inappropriate ways. She wrote in part (b):

> Depending on the case of the student, I might have started from making him identify his mistake by making him work in the same way but on a simpler exercise such as $|x\text{-}1| = 0 \Leftrightarrow (|x\text{-}1|)^2 = 0$ ... etc.

Candidate [25] expresses a preference for engaging the student in a process of self-realisation but she attempts to do so with an irrelevant example which doesn't help the student identify his[5] mistake: it can be solved without distinguishing cases and it has a solution. She adds: "To another student (in the case of a student who has the knowledge but likes to complicate otherwise easy things!) I would try to point out the obvious". This comment contains elements of reflection on student thinking and of meta-cognitive feedback to the student. However it is too vague to allow the good intentions to be transformed into useful, concrete suggestions. The candidate felt the need to use a problem or a simpler exercise as a pedagogical tool to help the student understand his error. But the choice of example is inappropriate, even though technically correct. It seems that she could not translate good pedagogical intent into a mathematically coherent and didactically effective suggestion.

We identified analogous tendencies in the scripts of candidates [45] and [50] who place great emphasis on the routine method and choose an example that is supposed to support the refutation of the student's error. Unfortunately their choices are based on an inaccurate judgement regarding the origin of the error (M1). In the occasion these mathematical inaccuracies stall the effect of their otherwise good pedagogical intentions (preference for a dialogue with the student; employment of examples). Finally candidates [23] and [43] devote almost the entirety of their response to procedural aspects of solving the inequality thus diverting students from the conceptual and meta-cognitive aspects of the exercise.

[5] We refer to the candidates as 'she' and to the student in question as 'he'.

PEDAGOGICAL SENSITIVITY TRAPPED AND DIVERTED?

The above scripts were produced by mathematics graduates with some teaching experience and with some interest in reflecting on pedagogical issues. In the cases we examined here briefly there is evidence of positive but stalled pedagogical intent. The main characteristics of the candidates' problematic subject-matter and pedagogical content knowledge were: insistence on standard procedural methods; inappropriate contextualisation of generally commendable pedagogical practices; and, inadequate or inaccurate reflection on student thinking. We highlighted especially how these may impede the didactical effectiveness of the candidates.

Twenty one responses were allocated in D2. Why do these candidates turn to these correct yet unnecessarily, in our case, convoluted approaches? Because their mathematical thinking lacks the flexibility and creativity suggested by less routine (perhaps instigated by mathematical training that is inadequate in this respect)? Because their thinking – mathematical and pedagogical – gravitates towards a deeply ingrained belief in standard, routine methods (perhaps instigated by extensive involvement with, generally, procedurally-based school mathematics)? In any case, despite 'knowing-about'[6] the subject, they are in need of a transition towards 'knowing-to' act.

> Knowing to act when the moment comes requires more than having accumulated knowledge-about. It requires relevant knowledge to come to the fore so that it can be acted upon; [it] requires awareness …[It] is working on this awareness which provides the fulcrum for professional development' (Mason and Spence, ibid, p139).

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[6] Knowing about: an overarching type of knowledge that covers Shulman's categories and includes knowing-that (factual knowledge), knowing-how (technique and skills) and knowing-why (explanatory and reconstructive fluency).

STRUCTURING STUDENTS' AWARENESS OF GENERALITY IN WHOLE CLASS DISCUSSION

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While leading whole class conversations, as a teacher, I aim to remain aware of students' powers to appreciate and express generality. In this paper I focus on a whole class conversation that took place during a game of 'algebra bingo'. I consider the various types of generalities and examine the extent to which students in the class might be aware of each generality.

INTRODUCTION

Much of my focus as a teacher, whether considering language use, task design, or lesson structure, centres on structuring awareness. I see it as my responsibility to help students focus their attention on those things that I think are particularly worth attending to. I have recently been concerned with the specific effect that using technical mathematical language can have on focussing students' attention on generality. Activities where skills are practised, such as 'algebra bingo', can be developed so that they also offer opportunities to explore generality. The discussion surrounding such practice activities may appear more relevant and interesting as a consequence of being triggered by such a game. In developing students' sense of generality, for example, their attention can be drawn to which numbers in an example are structural, and which are particular, or whether a statement is sometimes, always or never true.

When examples are used to illustrate a whole space of possibilities, it is important that students appreciate this general space, and are able to generalise. Krutetskii (1976) observed that the more successful mathematics students in his study were those who could generalise 'on the spot', on the basis of an analysis of just one phenomenon. Following this argument, it would seem desirable for mathematics classrooms to be places where all students become better able to see the general through the particular (Whitehead, 1932). The discussion of generality is thus a central part of mathematics education, and the role of the teacher in promoting and guiding such discussion is worth serious consideration.

ALGEBRA BINGO

7	6	9
11	15	1
2	12	3

The whole-class 'conversation' that follows took place during a game of 'algebra bingo'. Each student drew a 3-by-3 grid and filled it with their choice of integers from 1-15 inclusive (see diagram for example). At the start, students decided that a = 13 and g = 4. An algebraic expression was written on the board using these letters, and students calculated the value and crossed off the answer in their grids, if they had it. The winner was the first student to cross out all 9 of their numbers.

My first three questions were:

$2(a-8)$		10
$3(7-g)$	with corresponding answers of	9
$2(a-9)$		8

Gender-preserving pseudonyms have been used for each student. '//' indicates one speaker being interrupted by another.

1	Me:	All the questions are having brackets today. I'll start, let me know if you want to suggest a question.
2	Gina:	You're doing a pattern Miss. We'll stop thinking!
3	Me:	Oh yes. But if I carry on like that I'll have to do 7, which seems tricky.
4	Laura:	You could put 7 in front.
5	Sam:	You'll have to put 7 in front.
6	Me:	Will I?
7	Gina:	Or you could put 1, but//
8	Sam:	//That would be silly, you don't need to multiply it by 1.
9	Natalie:	Multiplying by 1 doesn't do anything.
10	Me:	$7(a$-$12)$ Does anyone want to make up a question?
11	Laura:	I want *a*. Can I have *a*?
12	Me:	No. I want brackets today.
13	Laura:	Ok. Put it in brackets! Can you do that?
14	Me:	[Writes (*a*)] I suppose it would mean the same as *a*, but it doesn't count.
15	Chris:	[To Laura] Put a 1 in front of the brackets.
16	Rebecca:	[To Laura] Do $(a-0)$ in the bracket.
17	Students:	[Talk amongst themselves. Some of the talk is relevant, some is not.]
18	Me:	Ok. [Pause for quiet]. Let's think about this together. How many different ways can we make 13?
19	Chris:	1, *a* in brackets.
20	Me:	[Writing $1(a)$]. What do you think?
21	Rebecca:	Aren't the brackets supposed to be for doing something? Like $a-0$?
22	Chris:	Or *a* times 0.
23	Me:	[Writing $1(a-0)$ and $1(a \text{ x } 0)$ on the board]. Are they both the same?
24	Laura:	*a* times 0 is 0.
25	Chris:	Oh yeah.
26	Me:	Are they both the same?
27	Chris:	No, my one doesn't work. $a + 0$ would though.

Several students then put their hands up to suggest other expressions, and the game continued.

ANALYSIS AND INTERPRETATION

Analysis of this transcript led to an increasing appreciation of a variety of categories and levels of generality contained within it. The generalities examined below are those that I can perceive in the discussion, but another reader may discern further examples, or dispute my selection. The same can be said for the students in the classroom, and the extent to which they are aware of the generality being expressed. Whilst a teacher striving to encourage students to think mathematically and to form generalisations may be pleased to note an increase in the 'presence' of such occurrences in their lessons, such generality needs to be present in some sense for the students as well as for the teacher.

Generalisations about… algebra

Students appear to be using general rules about algebra in the extract. Rather than ask for the number she wanted (13, in line 11), for example, Laura said "I want *a*. Can I have *a*?". This seems to be an application of the rule that a letter can represent a number. For her, it seems, for this game, *a* and 13 have become interchangeable. The understanding that a letter represents a number is considered to be an essential foundation of algebraic understanding (Rosnick and Clement, 1980).

There is some ambiguity about whether students are *using* a general rule in a particular case, or moving from the particular to *express* a general rule. Because the students are using algebra, some of their applications of generality may be interpreted as expressions. For example, when Chris and Rebecca suggest ways that Laura can achieve the answer *a*, their suggestions 1(*a)* and (*a* - 0) would equal *a* for *any* value of *a*. They may be saying more than that multiplying 13 by 1 gives 13, or that 13 – 0 is 13. We cannot tell whether they, or the other students, are aware of this. A listener could interpret their suggestions as being true only when $a = 13$.

In line 5, Sam suggests that an answer of 7 can only be achieved if a 7 is put in front of the brackets. With this distinction between 7 as *a possible* answer, and the *only* way to get 7 as an answer, Sam appears to demonstrate some general sense of 7 as a prime number. His expression could perhaps be interpreted as '7 is the only interesting factor of 7', which seems to be a more general statement than '7 is a factor of 7'. Natalie appears to be justifying Sam's particular statement about multiplying a bracketed expression by 1 to obtain the answer 7 with the expression of a general rule:

8	Sam:	That would be silly, you don't need to multiply it by 1.
9	Natalie:	Multiplying by 1 doesn't do anything.

With her use of the present tense here, Natalie seems to imply the general 'multiplying any number by 1 doesn't do anything' rather than the more particular 'multiplying this number by 1 wouldn't do anything'.

Uncertainty about what type of generality is being expressed, which I have experienced while analysing the transcript, might also be experienced by students trying to make sense of the discussion for themselves. They may also be uncertain about whether a general claim is true. With the intention of encouraging students to think about each others' conjectures, and to realise that the truth or falsity of mathematics can be determined without an external authority (the teacher), I tend not to correct students' imperfect conjectures immediately. A student referring to a general rule might, if given space and time, provide a prompt for other students to test the conjecture and develop their understanding. It would be debilitating, however, if deep consideration were required on every occasion a generality was implied or inferred. Although a student may well generalise the rule, and even verbalise the rule, for themselves, the teacher might promote the process by emphasising its importance. Some generalities will be more useful than others, and the teacher may be able to indicate to the student which these might be.

Generalisations about… the game

I write 'game' here, but few of the students appear to see competition as the main purpose of the activity. I find myself frustrated when the activity is interrupted by students telling us "I've got a full house!". Some of the students seem to feel the same. When someone won and we stopped, for example, Laura quietly said "I got a full house ages ago". She apparently didn't think it was interesting enough to tell us about at the time. It is tempting to conclude that the game is an irrelevance, but I suspect that it acts to focuses students' attention in the first place, prompting an initial act of engagement.

It is unclear how students distinguish between the rules of mathematics and the rules of the game. When Laura asks, in line 13, whether she can put *a* in brackets, it is unclear whether she is clarifying the rules of the game or of the mathematical world. Likewise, when Rebecca questioned whether you would or could have brackets that weren't for "doing something" (line 21) she may have been asking about the rules of conventional algebraic notation, or the rules of the game. As I had insisted (line 1) that all expressions must have brackets, Rebecca may have been interpreting my game rule as 'all expressions must have brackets that do something'. Her contribution can be seen either as a reminder of the rules of the game, or a general statement about the meaning and role of brackets in mathematics. In either case, her observation is a non-trivial one, especially given that in written language brackets are often used to designate the inessential.

It is possible to distinguish levels of convention or arbitrariness in mathematics, of which games, tasks and exercises are perhaps the most arbitrary and transient. Students' awareness of the level at which a generality is operating seems crucial to their understanding. The arbitrary nature of general rules in classroom activities often acts to limit the range of permissible change (Watson and Mason, 2005) associated with an aspect of mathematics. "Pick any three numbers", for example, may always mean (for a given group, with a given teacher) "pick any three 2-digit integers". If many activities require this sort of number, then time is saved by establishing 2-digit integers as the range of permissible values for these questions. If students understand that this is a classroom convention, and are still aware of the huge range of numbers that is actually available, then this practice is unproblematic. Unfortunately, this is unlikely to be the case.

I experienced the effect of falsely reducing the dimensions-of-possible-variation for myself during the lesson, when I expressed the difficulty of ‘making 7’ within the restriction ‘with brackets’. I was restricting my interpretation of ‘with brackets’ to only those expressions of the form $a(x + y)$. In retrospect, I could have introduced 2(g – 2) + 3, or an equivalent, thereby allowing for many more possibilities. None of the students suggested an expression of the form $a(x + y) + b$, which suggests that, for this activity at least, the dimensions of possible variation of an expression with brackets did not include such a form.

Line 10 might have been a good opportunity to ask “Could I put anything else, or just 7 and 1?”, and discuss prime numbers and factors. I could have let the students think of alternative ways to make 7, but I felt that the pace of the activity would suffer. Many classroom activities have an extra purpose, alongside that of learning a particular topic. With tests the focus can move from learning mathematics to ‘getting a good mark’, while with games it might move to ‘winning’. While these objectives are generally seen as less important than the mathematics, they may offer students an incentive to access the task.

Generalisations about… behaviour and purpose

Many of the students’ suggestions and assertions give insight into more than their knowledge of mathematics. Just as rules about mathematics can be formed on the basis of several particular examples (or even just one), rules about behaviour are being formed and revised based on particular instances in lessons. The general understandings of a group at a time combine to constitute a community of practice (Lave and Wenger, 1991) or a local community of practice (Winbourne and Watson, 1998). These generalities can be discerned from particular instances.

One such generality concerns the role of errors and misconceptions in maths lessons. Chris’s readiness to accept that “my one doesn’t work” (line 27) and to suggest an alternative is not exceptional in this group. Chris’s comment can be seen as a particular example of it being ‘safe’ to admit being wrong. The comment may be used by other students to create a general rule, leading to a conjecturing atmosphere.

A second generality that I can perceive in the discussion is the rule that ‘when other students are talking amongst themselves, it’s ok for us to do that too’. A large proportion of the students were offering questions, answers, or other contributions. They were otherwise silent. This is perhaps the nearest I can come, as an observer, to claiming that they were ‘listening’. The social contract that ensures that students listen during such discussions seems to break down in line 17. This particular occurrence is an example of a general tendency for students to talk amongst themselves if I appear to have taken a step back from the discussion. As their teacher, I might be able to share with them explicitly the objective that they listen carefully to discussions in which I am not participating.

This kind of meta-communication (communicating about how we communicate) seems to have had an impact in other areas. In line 2, for example, Gina appears to be discouraging me from making it too easy, as they will 'stop thinking'. In other lessons also, these students appear to see thinking as the central purpose of mathematics lessons. I believe that this is partly due to the emphasis that I placed, especially at the start of the year, on the value of thinking.

REFLECTIONS

The bingo game offers an illustration of how tasks with the principal objective of practising skills can provide an opportunity for the discussion and development of ideas. The extent to which students take up this opportunity is greatly shaped by the role played by the teacher. In just a few minutes of classroom discussion, generalities were expressed concerning algebra (cognitive generalisation), the rules of the game (social generalisation), and behaviour (behavioural and possibly affective generalisations) in mathematics lessons. If there is a value for students in being in the presence of these abstract ideas, then such a benefit might be increased by emphasising the generalities, and drawing students' attention to what is happening in the discussion. Ellis (2005) found that teacher requests for justification led to "more productive" generalisation, and that repeated use of "why" led to improved, higher-level restatements of the generality. Unless the teacher questions and probes for this justification, the possibility remains that surface-level observations will have the same status amongst listening students as those that are much more profound.

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Hewitt, D. (2006) Proceedings of the British Society for Research into Learning Mathematics 26(2) June 2006

PRIMARY TRAINEE TEACHERS' KNOWLEDGE OF PARALLELOGRAMS

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Considerable research has indicated that amongst the factors which make the most significant contribution to high student achievement in mathematics is secure subject knowledge on the part of the teacher as this underpins an approach to mathematics in which topics are seen as part of a coherent set of related ideas, with clear progression and links to previous and future learning. This paper reports part of the findings from a study of trainee teachers' knowledge of basic geometrical figures, in particular focusing on what knowledge they have of parallelograms and how they use this knowledge to solve geometrical problems. The findings indicate that only a minority of trainee primary teacher have a fully sophisticated knowledge of parallelograms and of how to use the properties of parallelograms to solve relevant problems.

INTRODUCTION

It is well known that teachers' mathematics knowledge plays a significant role in shaping the quality of their teaching (Ball, Hill & Bass, 2005). Yet as Ball *et al* (*ibid*, p16) explain, "although many studies demonstrate that teachers' mathematical knowledge helps support increased student achievement, the actual nature and extent of that knowledge - whether it is simply basic skills at the grades they teach, or complex and professionally-specific mathematical knowledge - is largely unknown". This is not to downplay the studies of teachers' mathematical knowledge that have been, and are being, carried out. More it points to the complexity of the issues involved, especially since the context in which teachers gain their own mathematical knowledge, and the form of teacher training they receive (both pre- and in-service), can be so varied, not only across countries, but also within particular countries.

The data reported in this paper are from one component of a larger study being carried out in the UK. The over-arching focus is on teachers' knowledge of geometry since, at this time in the UK, the nature of the school curriculum is under review (QCA, 2005) and there are recommendations that the geometry component of the mathematics curriculum requires special attention and strengthening (RS/JMC, 2001). The chosen focus for this report is on what knowledge trainee teachers have of parallelograms and how they utilise this knowledge when tackling geometrical problems.

CLASSIFICATION OF PARALLELOGRAMS

Mathematicians prefer a hierarchical classification for quadrilaterals (de Villers, 1994) and school curricula also follow this. One reason for this preference is its economical character; that is, if a statement is true for parallelograms, this means that it is also true for squares, rectangles and rhombuses. While this might seem straightforward to mathematicians, a number of studies have shown that many students have problems with a hierarchical classification of quadrilaterals (de Villers, 1994, p17; Jones, 2000).

Van Hiele’s model of the learning of geometry, which suggests that learners advance through levels of thought in geometry (Crowley, 1987, van Hiele, 1999), is generally considered to be a fairly useful model to describe students’ behaviours in geometry (Senk, 1989). The model specifies the following four levels: Level 1, visualisation: identifying shapes according to their concrete examples; Level 2, analysis: identifying shapes according to their properties; Level 3, informal deduction: identifying relationships between shapes & producing simple logical deduction; Level 4, deduction: understanding logical deduction.

In terms of this model, students at van Hiele level 3 are, for example, expected to be able to deduce that a rectangle is a special type of parallelogram by considering definitions and properties of these quadrilaterals. Students at level 2 start recognising properties of individual shapes (e.g. that in a square all the sides are the same and that all the angles are the same), while students at the level 1 would recognise a square or rectangle from their shape, and that they are different from a circle. Research evidence suggests that the rate of progress from level 2 to 3 made by many students is slow, or even that many of them remain at level 2 by the end of the secondary (high) schools (Senk, 1989). Thus, the hierarchical classification of quadrilaterals, taken to be van Hiele level 3, can be regarded as a difficult task for many children.

There is some existing evidence that trainee primary teachers have relatively weak knowledge of geometry (see, for example, Jones, Mooney and Harries, 2002). Data from observing such trainee teachers has also indicated that difficulty with the hierarchical classification of quadrilaterals appears to persist with trainee teachers as at least some of them cannot accept, for example, that ‘a square is a special type of a rectangle’ (Fujita & Jones, 2006). In this paper, we explore this latter issue in particular by considering the questions ‘what images of parallelograms do trainee teachers have, and how do they use them to solve a geometrical problem?’

METHODOLOGY

In order to investigate the research questions, trainee primary teachers on a four-year undergraduate teacher training course in Scotland were selected because the curriculum guidelines for Scotland specify that most pupils are expected to be able to define quadrilaterals and classify them in accordance with their properties by the time they are aged 14-15. What is more, the expected level of understanding of mathematics for trainees on the course is that, to be allowed to commence the course, trainee have to have a level of mathematics indicating that they are able to classify quadrilaterals according to their definitions and properties (in Scotland this is called ‘*Standard Grade Credit level*’).

We conducted a survey of 105 trainee primary teachers in their second year of University study (most were 19~20 years old) in February 2006. The questionnaire used is adapted from one originally designed to measure the level of understanding of parallelograms held by equivalent trainees in Japan, and it consists of six questions. Because of the limited space of this paper, we report a part of results relating to items 1, 3 and 5 of the questionnaire:

Q1. Choose which are parallelograms from the quadrilaterals 1~15 below.

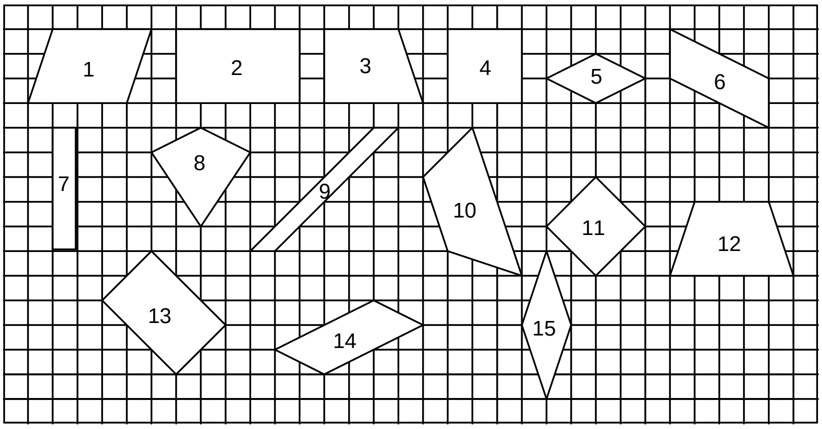


Figure 1. Quadrilaterals in Q1

Q3. Is it possible to draw a parallelogram whose four vertices are on the circumference of a circle? Choose your answer a. or b. If you choose (a), state your opinion why it is not possible. If your answer is (b), draw its shape and name in the circle. (a) No it is not possible, because … or (b) Yes, it is possible.

Q5. What is the quadrilateral described as 'a parallelogram which has a right angle'?

While Q1 checks students' basic knowledge of parallelograms, Q3 asks students to determine parallelograms which can be inscribed in a circle. The answer to Q3 is a rectangle. This question checks whether students can use a hierarchical relationship to solve a problem, i.e. whether they can understand that it is not only a square, but also any kind of rectangle, of which a square is just one, which can be inscribed in a circle. Similarly, Q5 checks whether they can decide the type of a parallelogram which will satisfies additional information (a right angle). The table below summarises the marking criteria for each question. For example, in Q1, if a student can identify parallelograms correctly (i.e. 1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 14, 15 in fig. 1), he/she receives '3' points for Q1.

Table 1 Marking criteria for Q1, Q3 & Q5

	3 pt.	2 pt.	1 pt.	0 pt.
Q1	1, 2, 4, 5, 6, 7, 9, 11, 13, 14, 15	1, 6, 9, 14, 5, 11, 15 or 1, 2, 4, 6, 7, 9, 11, 13, 14	1, 6, 9, 14	Others
Q3	b. and draws & name a rectangle	b. and draws & name a rectangle and a square	b. and draws a correct image b. and draws & name a rhombus	Others
Q5	Rectangle Rectangle (square)	Rectangle & square Square	Trapezium	Others

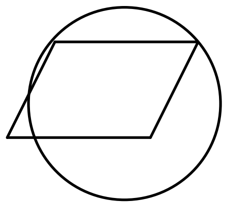
RESULT AND DISCUSSION

Table 2 summarises the results of the Q1, Q3 and Q5.

Table 2 The result of Q1 & Q3

	Q1	Q3	Q5
3 pt	21 (20%)	8 (7.6%)	45 (42.9%)
2 pt	27 (25.7%)	11 (10.5%)	23 (21.9%)
1 pt	47 (44.8%)	10 (9.5%)	0 (0%)
0 pt	10 (9.5%)	76 (72.4%)	37 (35.2%)

Figure 2. Q3 & Prototype image



As table 2 shows, while just 21 of 105 trainees (20%) could identify all correct images of parallelograms, 47 (44.8%) chose the images 1, 6, 9 and 14 (fig. 1) which are likely to be considered to be prototype images of parallelograms. This implies that almost half students still regard parallelograms in terms of prototype images. The performance by the trainees on Q3 shows their weaknesses with this topic. While just 8 trainees (7.6%) could answer that 'it is a rectangle which satisfies the Q3', 76 (72.4%) answered '(b) No, it is not possible'. Of these 76 trainees, 20 trainees gave no answer, 15 reasoned their answer by stating 'you will always get a right angle to draw a parallelogram in a circle' or 'There will only 2 vertices which touch', five just drew an image based on a prototype image of a parallelogram (fig. 2), and 20 stated their reasons and drew an image based on the prototype image, i.e. 40 of 76 trainees who scored '0' point in the Q3 used the prototype image of a parallelogram to tackle this question. The performance for Q5 is slightly better than the other questions, in that 45 trainees (42.9%) could answer correctly, i.e. they could answer that a parallelogram which has a right angle is a rectangle.

In Q1, about half of the trainees chose just prototype images of parallelograms, and, again, 40 used the prototype images to solve the Q3. There is a slight correlation between the performances of these questions. The value of the Pearson‘s correlation coefficient between Q1 and Q3 is $r=0.48$ ($p<0.01$), i.e. the trainees who could choose the correct images of parallelograms are likely to get better scores than those who just chose the prototype images. A reason for why there is not a strong correlation is that 6 of 21 trainees who scored ‘3’ points and 18 of 28 who scored ‘2’ points for Q1 could not answer correctly Q3, i.e. their ability to control their images still does not appear consolidated when they solve geometrical problems. In Q1, 27 trainees scored ‘2’ points, and 17 of them (about 63% of 27) chose the images of rectangles (e.g. 2, 7, 13 in fig. 1) or squares (4 or 11 in fig. 1), and the others chose the prototype images of parallelograms (1, 6, 9, 14 + 5, 9, 15 in fig. 1). Interestingly, 8 of 17 could get either ‘1’, ‘2’, or even ‘3’ points for Q3. This implies that one of important factors in solving Q3 is what images of parallelograms they would utilise.

The results from Q3 and Q5 somehow contradict each other. While only 18 could see ‘a rectangle (or a square)’ as ‘a parallelogram’, in Q5 more than half of them could answer ‘a parallelogram which has a right angle is a rectangle (or a square)’, i.e. they could see ‘a rectangle (or a square)’ as ‘a parallelogram’. A reason for this idiosyncratic result is still uncertain, but a possible reason is that the words ‘the quadrilateral’ and ‘a right angle’ in the question might remind them of ‘a rectangle (or a square)’, rather than them using geometrical reasoning to answer this question.

CONCLUDING COMMENT

The findings indicate that only a minority of trainee primary teacher have a fully sophisticated knowledge of parallelograms and of how to use the properties of parallelograms to solve relevant problems.

In terms of the van Hiele model, on the one hand, the results described above can be explained by the fact that the trainees are still at the level 2 (or below) and therefore they just choose only prototype images of parallelograms, and do not use geometrical reasoning when they solve Q3. On the other hand, to be allowed to commence their training course, the trainees have to have a mathematics qualification at a level that indicates that they should be able to classify quadrilaterals according to their definitions, i.e. the level 3 in the van Hiele model. The results described above suggest that regression has happened after they have achieved Scottish *Standard Grade Credit level* (at age 16). It is necessary to investigate why this regression might happen, in particular focusing on curriculum design, textbooks, actual teaching of geometry etc. As Ball *et al* (2005, p. 16) recommend “What is needed are more programs of research that complete the cycle, linking teachers’ mathematical preparation and knowledge to their students’ achievement”.

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OBSERVING SUBJECT KNOWLEDGE IN ACTION: CHARACTERISTICS OF LESSON OBSERVATION FEEDBACK GIVEN TO TRAINEES

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This paper outlines the main findings of a small-scale study in which the relative frequency of comments about elements of mathematics teaching found in written feedback given to primary trainee teachers by school-based mentors, university mathematics tutors and university tutors of other subjects was investigated. Further issues are raised regarding the awareness of some observer groups about 'contingency' elements of trainees' mathematics teaching.

INTRODUCTION

Shifts in Initial Teacher Training Requirements

'Teaching: High Status, High Standards' (DfEE, 1998) which required teacher training providers to audit and address perceived weaknesses in trainees' subject knowledge assumed a 'deficit model of teacher's knowledge: highlighting what they appear not to know and deducing that improving teachers' own subject knowledge would lead to better teaching' (Poulsen, 2001, p. 43). More recent requirements state that 'evidence of secure subject knowledge and understanding is most likely to be found in trainee teachers' teaching, particularly in how they present complex ideas, communicate subject knowledge, correct pupil errors and in how confidently they answer their subject-based questions' (TTA, 2002, p. 19). This shift in approach is based upon a new perception that subject knowledge is highly situated in classroom practice and can therefore be best acquired and assessed in this context.

Shifts in Research: The Nature of Knowledge

A similar shift exists in research-based literature. Much work on teacher subject knowledge has been based on Shulman's identification of three aspects of knowledge: subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge. Shulman suggested that teachers 'transform' their personal knowledge (SMK) into a form that is meaningful to learners using their pedagogical content knowledge (PCK) (Shulman, 1987, p. 15). More recent work suggests that teacher subject knowledge is 'pedagogically situated within the socio-cultural community of practice of primary school teaching' (Poulsen, 2001, p. 44), is grounded in, and constrained by, classroom experience, values and beliefs (Aubrey, 1996, Meredith, 1993) and is therefore more complex than Shulman's model. A framework developed by Huckstep, Rowland and Thwaites (2002, pp.5-6) for their videotape study of 24 lessons prepared and conducted by trainee primary school teachers represents a recent approach to categorizing teachers' mathematical knowledge as evidenced in the classroom. They identified 18 categories of mathematical knowledge which have been grouped within 4 broader 'dimensions' known as the 'Knowledge Quartet': '*foundation*, *transformation*, *connection and contingency*'. A subsequent pilot project (Ainley & Luntley, 2005) develops the last of these dimensions, *contingency*, further by suggesting that it is situated '*attention-based knowledge*' which enables teachers to respond effectively to what transpires within lessons and that their evaluations of the outcomes of their contingency-related actions inform their selection of future responses in such situations.

Research about Feedback given to Trainees

Even experienced teachers find it hard to articulate what it is that they do successfully in the classroom other than in highly situated accounts (Edwards & Collinson, 1995) so review meetings following lesson observations may be crucial opportunities for developing those aspects of trainees' practice which are subject knowledge-related. However, in a recent USA-based study, only 2% of oral lesson feedback was concerned with subject-related issues (Strong & Baron, 2004, pp. 52-53). For many trainees their school placement mathematics becomes 'completely subsumed by pedagogical concerns' (e.g. classroom management) and this may be why review meetings between school-based teacher-mentors and trainees tend to focus on generic pedagogical skills rather than mathematical aspects of lessons (Brown, McNamara, Hanley & Jones, 1999, p. 309).

METHODOLOGY

The contents of 57 lesson observation proformas completed by university mathematics tutors, university tutors of other subjects and school mentors (19 proformas from each of the 3 observer groups) during their routine observations of primary trainee teachers' mathematics lessons were analysed in order to identify different elements of subject knowledge-related practice that had been judged worthy of comment. The proformas selected related to trainees from different stages in initial teacher education programmes and from a range of such programmes. A range of different observers were included within each observer group.

The different comments made by observers were assigned to one or more of the 18 categories from the 'Knowledge Quartet' framework of Huckstep *et al.* (2002: 5-6) who regard this as a comprehensive 'framework for lesson observation and for teaching development' (Rowland, 2005). For this reason, this particular framework has been used as the basis for this study. Further categorization according to the type of observer followed in order to look for differences between the quantity of feedback offered by each observer group in relation to each element of subject knowledge-related practice. Those elements which were frequently or rarely mentioned in feedback were then identified. In total, 256 observer comments were assigned to one or more of the 18 categories. Thus, a notional 'average' category would attract approximately 14 (or 5·56%) of the total number of comments.

In addition, two school mentors and one mathematics tutor were interviewed (in a largely unstructured manner) in order to explore their perceptions about the nature of subject knowledge, how it was evidenced in trainees' teaching of primary school mathematics and which aspects they were likely to comment upon in their feedback.

FINDINGS AND DISCUSSION

High Frequency Elements

There was clearly a perception that overt subject knowledge should be treated as significant and this is probably unsurprising given the Teacher Training Agency's focus on this element in recent years. All of the interviewees dwelt at length on this. The high frequency of the other categories shown in Figure 1 probably reflects the observers' awareness of the strong emphasis given to these particular aspects of mathematics teaching by the National Numeracy Strategy.

Category	Mathematics Tutors	Other Tutors	School Mentors	Overall	% of Total
Overt Subject Knowledge	11	9	9	29	11·33%
Demonstration (Teacher)	10	12	7	29	11·33%
Identification of Errors	5	9	11	25	9·77%
Choice of Representation	8	6	10	24	9·38%
Use of Terminology	7	10	4	21	8·20%

Figure 1: High Frequency Elements

Low Frequency Elements

Figure 2 below indicates the categories which occurred least frequently. The very low frequency of comments about trainees' adherence to textbooks and concentration on procedures may be the result of university tutors discouraging this type of practice during university-based training. It is perhaps more likely, however, that all observer groups tend to accept these as common practices in schools and refrain from challenging them. The limited number of comments about theoretical underpinnings suggests that all the observer groups tend to treat the theoretical and the practical as disjoint elements of initial teacher training.

Mathematics is learned principally through engagement with examples (Watson and Mason, 2001) so trainees' selection of examples would appear to be an important aspect of their teaching to monitor. Indeed, Rowland, Thwaites and Huckstep (2003, p. 8) suggest that selection of examples appears to be a 'significant indicator' of mathematics content knowledge for teaching. However, in the content analysis undertaken very few observer comments were made about the trainees' choices of examples and none of the interviewees mentioned it. It would appear from the data that all of the observer groups fail to appreciate its importance.

Category	Mathematics Tutors	Other Tutors	School Mentors	Overall	% of Total
Concentration on Procedures	1	0	0	1	0·39
Adherence to Textbook	0	0	2	2	0·78
Deviation from Agenda	3	0	0	3	1·17
Theoretical Underpinning	1	1	1	3	1·17
Choice of Examples	1	1	2	4	1·56

Figure 2: Low Frequency Elements

The Contingency Dimension

Consideration of the distribution of observer comments revealed that little written feedback is provided to trainees about any of the three elements of the '*contingency*' dimension of the 'Knowledge Quartet (Huckstep *et al.*, 2002) (see Figure 3).

Category	Mathematics Tutors	Other Tutors	School Mentors	Overall	% of Total
Deviation from Agenda	3	0	0	3	1·17%
Responding to Children's Ideas	6	3	0	9	3·51%
Use of Opportunities	4	3	1	8	3·13%

Figure 3: The Contingency Dimension

Of the 20 comments made about the '*contingency*' dimension, 13 were made by university mathematics tutors. This suggests that other observer groups either attach less value to these aspects of teaching mathematics or are less aware of this dimension when observing trainees. In the interviews, only the mathematics tutor commented (favourably) upon these elements of a trainee's practice:

> Tutor: "The children made a tangential link and the student enabled them to make links to previous learning. It wasn't planned, just a response to something that the children said."

In contrast, the school mentors interviewed strongly associated sound subject knowledge with delivering planned lesson objectives, achieving learning intentions and adhering to schemes of work, an emphasis would seem to discourage the effective use of ad-hoc learning opportunities which may arise from children's responses since this would require some deviation from trainees' planning:

> Mentor 1: "Everything is objective-led isn't it? And unless they've got to grips with the Numeracy Framework and know what they are intending to get out of the lesson and what success criteria they are looking for ... then it doesn't matter how many elements of teaching they've got ... "

> Mentor 2: "I look for it in the planning, making sure that they are following the schemes

> of work the school has put in place, making sure that they are National Curriculum-led, ensuring that they are 'tied into' what the [class] teacher has done before".

This stance might also inhibit the development of trainees' ability to respond appropriately to unplanned learning opportunities which may arise in their lessons. Thus, the relative paucity of feedback provided by some observer groups in relation to the different elements of the '*contingency*' dimension appears to be a significant gap in the overall feedback provided.

CONCLUSIONS

The interviews undertaken indicate that written feedback given by observers of trainees' mathematics lessons may be affected by a number of factors such as the limited time available, observers' perceptions about relative importance of different aspects of teaching and the perceived need to be selective and to balance positive and negative comments. Moreover, the content analysis does not determine the quality of the written feedback offered about aspects of subject knowledge-related practice. Nor does it indicate the nature of any oral feedback which may be given in addition.

However, the data does indicate that certain key elements of trainees' teaching of mathematics (such as their choice of examples) are rarely mentioned in written feedback by any of the observer groups and these elements may benefit from more careful monitoring. Also, while university mathematics tutors seem to value and recognise the contingency-related elements of mathematics teaching when observing trainees' lessons it seems that school mentors and university tutors of subjects other than mathematics are less likely to do so and may even inhibit trainees' development in this respect. If Ainley and Luntley (2005) are correct in asserting that such '*attention-based knowledge*' is highly situated and is developed only through teachers' reflection upon the outcomes of their own responses to contingency-type events then it seems reasonable to conclude that these aspects of trainees' teaching need to be developed primarily within the school placement context. Thus, raising the awareness of university tutors of subjects other than mathematics and school mentors about this dimension of trainees' practice would appear to be advisable.

Additionally, it may be that an observer's ability to recognise the contingency dimension in action may also be highly situated and may only be developed through experience of observing mathematics teaching. This may perhaps explain why the majority of comments relating to the contingency dimension in this study were made by mathematics tutors and why the university mathematics tutor was the only interviewee to demonstrate some appreciation of this dimension. However, more research is needed to investigate this particular hypothesis further.

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RELATIONSHIPS WITH/IN PRIMARY MATHEMATICS: IDENTITY, EMOTION AND PROFESSIONAL DEVELOPMENT

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In this paper we explore primary teachers' emotional relationships with mathematics. Drawing on the concept of identity as a "defended self", we describe and analyse the case of one primary teacher. We argue that emotion as both an individual and a social element. Finally, we consider the implications for teacher education.

This discussion is based on an analysis of our interactions with a primary teacher, Ursula, as a researcher (Jeremy) and as a Higher Education teacher (Mike) as she participated in professional development in mathematics. At an earlier day conference, one of us discussed how Ursula's desire to be a mathematics teacher was a crucial factor in her professional learning (Hodgen, 2004). Here, we focus on how she became drawn to mathematics despite her initial avoidance of the subject and relate this to the social aspects of learning. Our central theme is a narrative that Ursula told and re-told during these interviews concerning Ursula's relationships with school mathematics and mathematics teachers, one of whom was Mike.

METHODOLOGY

Ursula was a teacher-researcher in the Primary Cognitive Acceleration in Mathematics Education (CAME) Project between November 1997 and July 2001 (Johnson, Hodgen, & Adhami, 2004). [1] As part of this project, Jeremy studied Ursula's professional development (alongside that of five other teacher-researchers) (Hodgen & Johnson, 2004). Eighteen months previous to this, she had attended a 20 days mathematics course that had been taught by Mike. [2] The data that we analyse is drawn partly from interviews conducted with Ursula as part of Jeremy's research study and partly from Mike's reflections on teaching Ursula. [3]

THE RESEARCH CONTEXT: URSULA AND MATHEMATICS TEACHING

Our interest in Ursula is as a "telling" rather than a "typical" case (Mitchell, 1984). When we first met her, Ursula was a primary class teacher. Later, in the second year of the Primary CAME Project, she became a Numeracy Consultant. In many respects, Ursula's professional development in mathematics was somewhat unusual for a primary teacher. Primary CAME was exceptionally extended and intense, involving one day release every fortnight for three years. Moreover, as part of the project, she co-wrote curriculum materials and gave research presentations to teachers and academics. She was also unusual in that she moved from a position of fear and distrust in mathematics to one in which she identified herself positively as a teacher of mathematics:

> I like being a specialist. I like having one subject. I like being a maths teacher.

Our analysis suggests, however, that this shift was due to a combination of factors not simply (or even mainly) to Primary CAME despite its intensity. Ursula herself identified three key moments in her own professional development as a mathematics teacher: a classroom incident in her first year of teaching when "something clicked" and she realised that she "might actually like teaching maths", the 20 days course and Primary CAME.

IDENTITY AND PROFESSIONAL CHANGE

Central to our analysis is the concept of identity. Becoming a teacher is a process in which identity is both the product and the instrument of change:

> identity … [is a] … culturally and personally located social schema that has the potential to be transacted, redefined and resisted and, like discourse, called upon when the moment is - to the learner - opportune. (Carr, 2001p.527).

But identity can be (and often is) a barrier to change. Bartholomew (2006) argues that identity is both a meaning making and a *defended* subject drawing on Hollway and Jefferson's (2000) conception of the "defended self", itself a development of Klein's work about "how the self is forged out of unconscious defences against anxiety" (p. 19). In her four-year study of involving 8 schools, Bartholomew argues that many of the teachers had forged an identity in terms of their ability to cope in a "difficult school" with "difficult students". This identification with "difficulty" meant that there were risks attached to success as well as failure. A successful school could no longer be a difficult school. Hence, these teachers had a very significant investment in the status quo. She uses the metaphor of the "dragon" to emphasise the power of these teachers' fear of change and argues that this fear of change cannot be removed, but rather needs to be transformed into a source of strength. As we have argued previously, this notion of the transformation of fear and anxiety into desire, or the attraction for something which "simultaneously attracts and repels", was key to Ursula's professional change (Hodgen, 2004). For Ursula, the dragon was strongly associated with her experiences of school mathematics and one particular secondary school teacher, Miss Briggs.

URSULA'S STORY

Ursula had what she described as a fear of equations, which she described as follows:

> I have a fear of algebra. […] No, I don't, I have a fear of anything that looks like an equation. And when people use the phrase simultaneous equations I could pass out quite happily. […] As, Bs, Xs and Ys, all mixed together, put in a few brackets and I'm gone, I'm away with the fairies.

Ursula had, however, been identified as mathematically able at school. She was in a small top set group who took an O/A in mathematics at the same time as GCSE. The O/A was taught in after school sessions by Miss Briggs (a pseudonym):

> that [the O/A] was after school, and that was with Miss Briggs. And Miss Briggs was actually, in retrospect, I think she was probably a very good teacher, because she did try

> to stand at the front and teach us. But I remember going from what I considered to be things that were really easy, drill and practice things, to this lesson where there was just this enormous algebraic equation going across the board. Absolutely enormous, and I'd walked in late because it was after school and I wasn't really over-keen on this, and I'd missed the first lesson, and I couldn't come to grips with this at all. And that was that. Walked out the classroom and didn't go back again. […] Supposed to work something out from it and everybody else in the room seemed to be able to do it except me. Because I'd kind of missed out on what I was meant to be doing with it. […] In fact, I could quite happily probably tell you for the last two years I probably didn't go to any maths lessons because I can't remember them at all, apart from the after school one. […] The only good thing about it was I kind of looked around the room and thought - mm, this is an interesting group to be in. But that's it. […] Yeah, there were some exceptional high fliers in that. It was a very small group. There must have only been about six of us. But most of them were the exceptional high fliers that just don't bear thinking about, you know, in your school year, they don't exist. They just get everything right, and you assume they do, and they don't exist as people. [LAUGHS.] But, only two girls and one of them was Susan. […] But I can remember the room as well. I can remember the room in minute detail. It obviously had huge impact because I could remember everything about it. […] And knowing how well I didn't get on with Miss Briggs it's quite likely that she made some derogatory comment that I didn't understand. Although I don't remember her doing it. It wouldn't have surprised me if she did.

There are a range of fascinating and important issues in this account that resonate with the literature - not least the construction of mathematical ability and failure (Boaler, Wiliam, & Brown, 2000). For our current purposes, and given space constraints, we simply highlight Ursula's sudden incomprehension of mathematics in the form of the "enormous algebraic equation" and the personalisation of this with Miss Briggs. This was a story Ursula told several times and on each occasion she immediately went on to relate an incident on the 20 days course in which she got frustrated with Mike about doing "maths for maths sake":

> The other person who got me very frustrated because I didn't understand was Mike. […] I obviously hadn't been listening and didn't know where it came from, and he was asking people to work out … about tangents … and there was something about it. But again it was using the vocabulary that I couldn't remember quite. And I got really cross with him and I kept saying - why do you want to put a line on a circle? For God's sake, why does anybody want to work this out with a line on a circle? I couldn't understand why anybody wanted to do it. […] You had to work something out around this, but I don't know what you had to work out because I'd given up by that point. And I just got cross with him. But again, I think it was into the realms of maths for maths sake. […] I just remember being very frustrated and feeling like I could have quite happily just sat there and cried, or screamed at him, one or the other. So I shouted at him instead.

MIKE'S STORY

Mike remembers the incident differently but equally clearly. In his recollection, the incident was concerned with negative integer powers and Ursula challenged rather than shouted at him. But, like Ursula, the focus is on the motivation for doing mathematics:

> Ursula, called me over.
>
> "My 5 quid calculator has got a 1/x button on, so why doesn't this expensive one?"
>
> "Well, it does," I replied " it's that x^{-1} button."
>
> This seemed an opportunity to explore powers and so I stopped everyone to go through the argument as to why, for consistency, mathematicians had decided to define x^{-1} as 1/x.
>
> While the teachers' nods during my explanation suggested that they were following me in the logic, afterwards there was quite a lot of muttering going on at Ursula's table.
>
> "Is there anything you are not clear about?' I enquired.
>
> "No, we follow your argument," Ursula replied. "But we were just saying to each other, 'why would anyone ever want to do that in the first place?'"

DISCUSSION

We emphasise that the 20 days course was a critical moment in Ursula's professional development. She described it as follows:

> it was just a good, a good course. It was a very inspirational. […] this is good maths practice, here some ideas you can go back and try. […] But it was all about, it was maths at adult level, I suppose, and maths at children level, […] He [Mike] did it well, 'cos we, we went back and we tried things but we also wanted to do things. So we were doing things on the train on the way home and we were phoning each other up and comparing how things had gone

Yet, one key incident for her - perhaps the key incident - was her frustration with Mike. Moreover, this incident seemed to be associated with her school experience with Miss Briggs. We suggest that, unwittingly, Mike provided a space within which Ursula could challenge the dragon of Miss Briggs and raise issues of purpose and motivation for doing mathematics. [4] We further suggest that this in turn enabled Ursula to begin to understand the attraction of mathematics (Hodgen, 2004).

Ursula's negative experience of secondary school mathematics is commonplace amongst primary teachers (Bibby, 1999). Indeed, we suggest that many primary teachers, like Ursula, have a dragon such as Miss Briggs. This we contend can lead to teachers protecting pupils from - or *defending* them against - mathematics. Such a position is likely to reproduce negative attitudes to mathematics.

It is commonly assumed that individuals are either drawn to or avoid mathematics. This pleasure/pain dimension is documented by Buxton (1981) amongst others:

> There seem to be two different states that one might properly call panic, yet they are *very* different. One is a sort of turbulence in the mind, a type of frenzy … 'mind chaos'…. More common in the maths classroom is a sense of paralysis, a freezing of the mind…. People may pass through the chaotic to the frozen stage or may simply enter paralysis directly. (p. 5)

More recently attention has been turned to the social dimension of learning mathematics. Bibby (2002), for example, describes how many primary teachers' relationships with mathematics are characterised by shame. Scheff (1994) describes shame as follows:

> Shame seems to arise from our need to feel the right degree of *connectedness* with others. Shame is the emotion that occurs when we feel too close or too far from others. When too close we feel exposed or violated; when too far, we feel invisible or rejected. Pride is the signal of being at the right distance: close enough to feel noticed but not so close as to feel threatened. (p. 40).

In our view, these emotional dimensions are dialectically inter-connected. As in Ursula's case, the motivation to do mathematics – or to teach mathematics – is both individual and social. Professional development in primary mathematics in the UK has generally focused on cognitive and pedagogic issues: teachers' mathematics subject knowledge, how children learn and teaching approaches. These issues are, of course, important. Indeed, our own professional development initiatives have focused on these issues. But, such an approach is, in our view, doomed to failure unless placed within an affective frame in which teachers have space to question mathematics and mathematics teaching. Central to such an approach is Noddings' (1992) notion of care – care for the learner and for the discipline. To paraphrase Noddings:

> Education […] has one main goal, a goal that guides the establishment and priority of all others, it should be to promote the growth of students [and teachers] as healthy, competent, moral people. […] We cannot ignore our children [and our teachers] – their purposes, anxieties, and relationships – in the service of making them more competent in academic skills.

NOTES

1. The Primary CAME Project was funded by The Leverhulme Trust as part of the 5 year Leverhulme Numeracy Research Programme (LNRP) at King's College London.

2. Ursula is a pseudonym. She gave permission for this data to be shared with Mike.

3. The methodology is described in detail in Hodgen (2003). The data is taken from interviews conducted on 4/3/99 and 19/7/00.

4. We emphasise that this was unplanned. Whilst we have thought for some time that emotion is an important aspect in primary teachers' mathematics learning, Ursula's case has caused us to radically re-think the nature of this.

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Hewitt, D. (2006) Proceedings of the British Society for Research into Learning Mathematics 26(2) June 2006

IS IT EVER APPROPRIATE TO JUDGE AN ARGUMENT BY ITS AUTHOR?

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There is a widespread belief in the mathematics education community that students should be encouraged to avoid basing their level of conviction in mathematical arguments on the authority of the argument's source. In this paper we report an experiment which investigated the role that authority plays in the argument evaluation strategies of undergraduate students and research active mathematicians. Our data show that both groups were more persuaded by an argument if it came from an authority figure. The implications of this finding are discussed. It is argued that the role of authority in mathematical argumentation – both in terms of actual behaviour and of normative behaviour – requires deeper scrutiny.

AUTHORITY IN ARGUMENTATION

In day-to-day reasoning it is well known that the source of an argument has a significant impact on how persuaded an audience will be. For example, when judging the persuasiveness of an argument about Avian Bird Flu, an audience is likely to react differently if it comes from a government scientist compared to a taxi driver. In mathematics, however, it is often claimed that the source of a proof should be ignored. Selden and Selden (2003), for example, wrote:

> "Like midcentury structural critics, mathematicians seem to treat a proof as being independent from its author." (Selden & Selden, 2003, p.6).

As part of their 'proof scheme' framework, Harel and Sowder (1998) defined a person's proof scheme as that which "constitutes ascertaining and persuading for that person" (Harel & Sowder, 2005, p.33). In a large study looking at what types of proof schemes students have, they found widespread evidence of the so-called *authoritarian proof scheme.* A person with such a scheme becomes persuaded of a statement's truth or falsity based on the pronouncements of an authority figure, often a teacher or textbook. Harel and Sowder (2005, p.42) described the authoritarian proof scheme as "an undesirable, yet common, way of thinking" and argued that "[instruction] must institute a didactical contract that attempts to suppress the authoritarian proof scheme" (Harel & Sowder, in press).

These negative comments about the authoritarian proof scheme seem to be based upon *a priori* views of expert practice which have not been verified by empirical research. In this paper we report a brief experimental study which empirically investigates how the presence of an authority figure influences conviction in mathematical argumentation. However, it is important to emphasise that, in this paper, we are not investigating proof in the sense of Selden and Selden (2003). Instead we advocate a broader framework for studying argumentation in mathematics that includes both formal proof and less rigorous arguments (e.g. Inglis & Mejia-

Ramos, 2006; Inglis, Mejia-Ramos & Simpson, submitted). Crucially, the experimental instrument discussed in this paper does *not* involve an argument that carries *absolute* conviction, and so direct comparisons with Harel and Sowder's work are potentially misleading. Notwithstanding this caveat, we believe that studying the factors which influence experts' conviction in mathematical arguments does have implications for the received view of the authoritarian proof scheme.

A CONJECTURE, AN ARGUMENT, AND AN EXPERT

Materials

Our task consisted of the following conjecture, argument and question:

> *Here is an open conjecture:*
>
> **Conjecture.** Somewhere in the decimal expansion of π there are one million sevens in a row.
>
> *Here is a heuristic argument about the claim:*
>
> All the evidence is that there is nothing very systematic about the sequence of digits of π. Indeed, they seem to behave much as they would if you just chose a sequence of random digits between 0 to 9. This hunch sounds vague, but it can be made precise as follows: there are various tests that statisticians perform on sequences to see whether they are likely to have been generated randomly, and it looks very much as though the sequences of digits of π would pass these tests. Certainly the first few million do. One obvious test is to see whether any short sequence of digits, such as 137, occurs with about the right frequency in the long term. In the case of the string 137 one would expect it to crop up about 1/1000th of the time in the decimal expansion of π.
>
> Experience strongly suggests that short sequences in the decimal expansion of the irrational numbers that crop up in nature, such as π, e or √2, do occur with the correct frequencies. And if that is so, then we would expect a million sevens in the decimal expansion of π about $10^{-1000000}$ of the time – and it is of course, no surprise, that we will not actually be able to check that directly. And yet, the argument that it does eventually occur, while not a proof, is pretty convincing.
>
> *After having read this argument please say to what extent you are persuaded by it:*
>
> not persuaded 1 2 3 4 5 totally persuaded

The argument and conjecture was taken from a book chapter written by Professor Timothy Gowers, a highly respected Fields Medal winning mathematician from the University of Cambridge (Gowers, 2006).

Method

We asked participants to complete the task above. Each participant was randomly assigned into one of two groups, the anonymous group and the named group. The anonymous group's task contained the line

> *Here is a heuristic argument about the claim.*

Whereas the named group's task contained the line:

> *Here is a heuristic argument about the claim (taken from a talk by Prof. Timothy Gowers, University of Cambridge).*

Participants completed the task online. The undergraduate students (*N*=302) were all from a single highly-rated university and were asked to participate by means of an email from the departmental secretary. The email explained the task and asked them to click through to the experimental website should they wish to participate. The research active mathematicians were recruited in two different ways. Some (*N*=14) were staff members at the same university as the undergraduates and were recruited in a similar manner. Others (*N*=60) were recruited through an advertisement posted on a mathematics research newsgroup. Before taking the task, participants in the researcher group were asked to self-declare that they were research active mathematicians.

The number of psychological studies conducted using the internet has dramatically increased in recent years, and the method obviously raises important questions regarding reliability and, in particular, possible multiple-submission. To prevent this, participants' IP addresses were recorded and analysed to detect possible rapid resubmissions. Space constraints prevent a full discussion of the reliability and validity of internet research here, although see Gosling, Vazire, Srivastava, & John, (2004) or Reips (2000).

Participants' comments on the argument

After participants had rated their level of persuasion in the argument, they were given the opportunity to write comments to explain their selection. The comments from the research active mathematicians emphasised the lack of agreement about how to react to the argument. The following selection of comments are typical of the range:

> Purely logically on the basis of the evidence presented, I am not persuaded at all. (Researcher, Anonymous Group, Rated 2).

> Normalcy of (the digits) of π is not unreasonable given almost all reals are normal. Then again, almost all real numbers don't have names, so who knows… (Researcher, Anonymous Group, Rated 2).

> It is persuasive but not a proof, but it only claims to be heuristic. (Researcher, Anonymous Group, Rated 5).

> The argument does not in any way claim to be a proof of the conjecture. […] The argument is actually for a much stronger result than the conjecture, and the conjecture could be true even if the argument is false. On balance, then, I would bet money that the conjecture is true. (Researcher, Named Group, Rated 4).

Along with there being no agreement as to whether the argument was persuasive or non-persuasive, a clear majority of both the undergraduates and the researchers rated their level of persuasion as being neither totally persuaded nor totally unpersuaded (in the range 2-4). This observation supports Inglis et al.'s (submitted) assertion that a key part of developing as a mathematician is the construction and appropriate qualification of arguments with non-deductive warrants.

Results

The mean level of persuasion from each of the groups are shown in Figure 1.

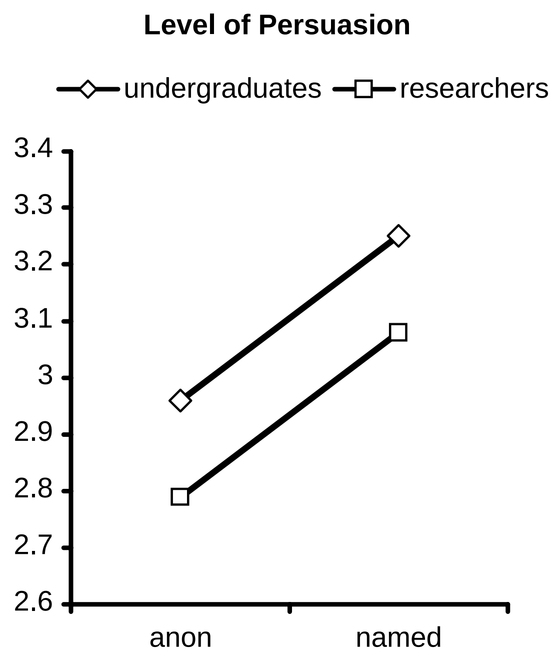

Figure 1: The mean levels of persuasion for each group for each version.

We conducted two nonparametric between-groups comparisons to identify possible main effects. There was no significant effect for group-type (undergraduate vs. researcher), Mann Whitney U=14052, *NS*. There was, however, a significant main effect for task-type (anon vs. named), Mann Whitney U=17498, p=0.014, with the group who knew the author's identity ranking the argument as more persuasive than the anonymous group.

Figure 1 clearly indicates that there was no task×group interaction, but to formally test for this we conducted an ordinal logistic regression analysis, resulting in a ordered logit model. The model was a poor fit, R^2=0.02, but the overall relationship was significant, $\chi^2(6)$=18.4, p=0.031. As the Mann Whitney tests suggested, in the model there was a significant effect for task-type, $\chi^2(1)$=3.95 p=0.041, and not for group-type, $\chi^2(1)$=0.347, *NS*. As expected, no evidence was found of a significant task×group interaction, $\chi^2(1)$=0.011, *NS*.

The results of these analyses are clear: participants in the study were more likely rate the argument as being persuasive if they were aware that it came from an authority figure. Furthermore, this effect was found in both undergraduates and research active mathematicians.

DISCUSSION

The data from this study indicate that when evaluating their level of conviction in non-deductive arguments, both undergraduates and research active mathematicians are influenced by the presence of an authority figure. Given the comments of Harel and Sowder (1998, 2005, in press) reported above, this result is perhaps surprising. However, these authors were referring to arguments which carry absolute conviction, not non-absolute arguments of the type used here.

There are also several potential differences between the argument used in this paper and a traditional proof. For one, some parts of Gower's argument need to be taken on trust. He asserts that statisticians have performed various tests for normality, but does not quote references or test outcomes. Perhaps participants who were not aware of the author's identity would be more likely to disregard this factual information as untrustworthy than those who knew Gower's status. Another possibility is that participants who were aware of the author's reputation would be reassured that he was not trying to hide evidence which would weigh against his argument, indeed one participant explicitly stated how knowledge of the author reassured him/her of this:

> We are told the argument is made by a reputable mathematician, so we implicitly assume that he would tell us if he knew of any evidence or convincing arguments to the contrary. (Researcher, Named Group, Rated 4).

However other participants made comments which suggested that the way they react to an argument (including proofs) does indeed vary depending on the identity of the author. One researcher commented that if the claim being made is sensible ("ordinary-looking") they would be less likely to devote time to checking the claim's justification if it came from a reputable source than they would if the author was unknown or unfamiliar. Ecologically this makes a lot of sense, disregarding the source of an argument when evaluating one's level of conviction in it involves giving up a lot of potentially useful information. Arguments proposed by reputable sources are more likely to be sound than those proposed by people with a history of putting forward erroneous claims. Throwing away helpful data when judging such matters would, in most domains, be regarded as irrational.

Notwithstanding the differences we acknowledge between the task used in this paper and formal mathematical proof, and between arguments that provide an absolute level of conviction and those that provide intermediate levels of conviction, we suggest that the data reported here indicates that the role of authority in mathematical argumentation (including proof) requires deeper scrutiny. Within a framework that models degrees of conviction in argumentation appropriate normative standards for the role that authority should play are unclear. In some cases it seems entirely reasonable to base conviction in a mathematical claim on an authority figure, when applying a published theorem to a new situation, for example; in other situations it seems less reasonable: most educators would not want to encourage their students to base their levels of conviction simply on the word of the teacher.

We suggest that existing descriptions of the role of authority in mathematical argumentation (and proof) are too imprecise. Harel and Sowder (1998, 2005, in press) suggest that basing conviction on authority is “undesirable” as it is not how professional mathematicians behave. This paper demonstrates that this need not be the case for non-deductive arguments. We suggest that the issue of whether it is the case for deductive proof remains an open question, one suitable for further empirical research.

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Hewitt, D. (2006) Proceedings of the British Society for Research into Learning Mathematics 26(2) June 2006

LEARNERS' SHIFTING PERCEPTIONS OF RANDOMNESS

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In clinical interviews, learners were invited to talk about their experiences of making sense of the emerging sequence of outcomes from repeated trials using different generators, some of which were biased. Analysis of the interviews revealed distinct ways of viewing the phenomena represented by the interview tasks. Drawing upon the local and global meanings of randomness identified by Pratt (1998), learners were found to shift rapidly between local and global perspectives. In this paper, data from a single interview is presented to illustrate the shifting perspectives.

STIMULUS TASKS

When examining people's perceptions and understanding of randomness, a key distinction to make is that between a random process and randomness in a sequence of outcomes generated by a random process (Zabell, 1992). Previous research into perceptions of randomness has not always been clear about this distinction (Nickerson, 2002), but some writers have discussed the issue explicitly (Falk and Konold, 1997; Wagenaar, 1991). The position that randomness is a property of a process rather than of the outcomes was adopted by Wagenaar (1991), and related to this is the view that any outcome from a random process is considered a random outcome (Pollatsek and Konold, 1991). However, it is only by observing outcomes from a process that one can judge whether the process is random.

Since my own introduction to statistics and probability at secondary school I have seen randomness as 'dynamic'. For me, a random sequence could not be printed out as a permanent record without losing the essence of what it is to be random. I felt strongly that random number tables were the antithesis of what I understood by "random". The sequence of numbers was the same whenever I opened the book, and I felt a need to invent a 'random' (and dynamic) process for selecting numbers from the page.

I see randomness as a model to describe a process, and as providing explanation for the outcomes observed. The sequence of observed outcomes might be described as 'random' if it were considered to have arisen from a process that is reasonably considered to be modelled by 'randomness'. This is a key aspect of the notion of randomness that I wish to convey to my students.

Commonly used stimulus tasks in previous research may be classified into two categories (Falk and Konold, 1997). Generation tasks require subjects to make up random sequences of outcomes to simulate the outcomes from a random process such as 'tossing a coin'. In recognition (perception or judgement) tasks, subjects decide whether a given outcome was produced by a random process, or select the 'most random' of several sets of results.

Falk and Konold suggest (1997) that recognition tasks "may be more appropriate for revealing subjective concepts of randomness" because "a person could perceive randomness 'accurately' and still be unable to reproduce it" (p302). Indeed there is clear suggestion from many studies (Nickerson, 2002; Shaughnessy, 1992) that people are generally not good at generating random sequences. Typically, people trying to simulate a random process tend to produce fewer long runs, and more alternations between outcomes, than would be expected from a random process (Falk and Konold, 1997). Other studies have used recognition tasks to explore what sequences people consider to be maximally random; these again show that people tend to identify randomness with sequences having an excess of alternations between outcomes (Falk and Konold, 1997).

In recognition tasks, the sequences presented are static, just as a random number table was 'fixed' for me. A person's attention is focussed only on a given sequence of outcomes and they may never consider the process by which these outcomes were generated. Generation tasks encourage a more dynamic view, since the subject may need to consider unpredictability as they 'generate' the sequence, and may be able to recall after the event the assumptions they had about the process. However, tasks requiring real-time interaction with outcomes from a random process would more effectively prompt learners to express their ideas of randomness as process.

METHOD

Eighteen learners, aged from 13 to 17 years, undertook clinical interviews for about an hour each. The study took place in two stages, with nine interviews in each stage. In the first stage, interviewees worked on tasks using three unusual dice: biased, spherical and cracked. In the second stage a new task using sampling bags was introduced. Interviewees were encouraged to talk reflectively about their experiences of working on the tasks and about what they were thinking.

The biased die looks like a standard cube, except it has two faces labelled 5 and no 3. It has a weight in the face labelled 1, so it is heavily biased towards showing 6. The spherical die is a hollow sphere containing a small bead. The sphere is marked symmetrically with numbers 1 to 6 and, when rolled on a flat surface, it always stops with one of the six numbers uppermost. If it is correctly balanced, each of the six outcomes should be equally likely. The cracked cubical die has a split running across the face labelled 6, and spreading partway across the faces labelled 2 and 5. Since interviewees used this die after their experiment with the biased die, I expected them to consider that it might also be biased.

In each experiment, the learner was first asked to comment on the appearance of the die and to consider how it might behave when rolled several times. The learner was invited to roll the die a few times before commenting on the observed outcomes. I encouraged learners to talk about their thinking throughout and I watched their behaviour closely. If a learner appeared to show concern about a run of outcomes, or even an individual outcome, I invited them to explain what they were thinking.

I hoped that using three different dice would increase the learner's awareness of what they expected from each die, and their willingness to articulate their assumptions. In particular, I hoped that the tasks would provoke learners to talk about how to recognise equally likely outcomes and whether these were necessary for the die to be considered to behave 'randomly'.

LOCAL AND GLOBAL PERSPECTIVES ON RANDOMNESS

Pratt's study (1998) of ways in which children aged 10 and 11 years articulated their ideas and beliefs as they worked in a carefully designed computer-based domain, distinguished two categories of meaning for randomness expressed by children. Local meanings were related to uncertain behaviour of the process and were focused on "trial by trial variation", while global meanings evolved as children recognised the importance of observing a larger number of trials and discerned features of distribution in the long run. Pratt saw the transition from local to global meanings in an individual as lengthy and complex. He did not report movement from global to local meanings, and it seems implicit in the tasks he set and the probes he used, that no such switch was either anticipated or looked for.

Analysis of interview transcripts in the present study showed learners thinking about random outcomes using two contrasting perspectives: the local and the global. In the local perspective, attention is on the uncertainty of the next outcome and ephemeral patterns that appear in short sequences of outcomes. The learner does not aggregate outcomes or to think in terms of a distribution. In the global perspective, the learner is aware of a distribution of outcomes, either empirically, as an emerging frequency distribution of observed outcomes, or in terms of prior beliefs about the generating process (for example, when rolling a die, expecting the outcomes to be equally likely). Learners' attention was found to shift frequently, and sometimes rapidly, between these perspectives, from local to global and back to local.

THE DATA

In this paper, I illustrate the rapid shifting between the local and global perspectives using data from a single dice task with the spherical die, in one interview with Ben from the first stage.

In the first seven throws, Ben had observed {5 5 1 4 1 6 1}. His initial strategy was to look for patterns in the sequence.

Ben: …we haven't had any 3s or 2s, so it could be one of those, but – well, it'll probably be another number than a 1.

I: Why?

Ben: Just from following the pattern. If it wasn't a die, that's what I'd say.

Ben noted that, since this was a die, any outcome was possible. He went on to look for an explanation for the absence of 2s and 3s.

Ben: It might be the way I'm throwing it though. Or when I picked it up, I'm throwing it the *same way*. *Or it could just be chance.*

Ben was thinking about the generating process, but still from a local perspective. Out of concern about lack of 2s and 3s, Ben went on to check the labelling of the die. When the fourteenth outcome was 3, Ben cheered!

Ben now attended to physical factors affecting the outcomes. He played deliberately with the die between rolls, and considered the shape of the weight moving inside the sphere. When I asked how Ben would know this was a fair die, he expressed clearly a global perspective based on his prior belief about a fair die.

Ben: …You just have to keep rolling it. It should in the end even out if it's a fair dice. **If it's not a fair dice it'll… keep on staying away from the 2s and 3s**, like it is at the moment.

I: Are you worried about it being fair?

Ben: …No, not really. …**It could just be chance**. If there's a 1 in 6 chance of getting each different number... I just haven't got a 2 yet, which is strange. Although **I'll probably get a 2 now**, if I roll it...

Ben appeared to be rapidly switching between contrasting views about this die. He understood he needed more trials, a characteristic of the global empirical perspective, and he expressed a prior belief about distribution: the die would be fair.

At the local level, he looked for the first occurrence of a 2, and expressed concern that he had not seen it after fifteen throws. At the global level, looking for a frequency distribution to match his prior belief, he accepted "It could just be chance". By changing the focus of his awareness, Ben arrived at two different explanations for the absence of 2.

After 17 throws without a 2, Ben had observed {5 5 1 4 1 6 1 1 5 4 4 5 1 3 6 5 4 5 5}. He was quiet, and he experimented with the die, rolling it in his hand without talking for 13 seconds, before commenting

Ben: It seems pretty fair. But it depends what happens when you roll it

He seemed to experience a tension between apparent 'fairness' of the process, and imbalance in the outcomes.

On the next throw, Ben rolled a 6, but he wanted the die to show a two. It was as though he wanted to remove the anticipation of waiting for a two to occur, and by experimenting with the way he rolled the die he was trying to make it happen.

Ben: …oh land on a 2.

On the next throw Ben rolled the die, and got a 2! He was excited and began to think he could control the outcomes. His experimenting with how to roll the die was associated in time with rolling a two. This supported his idea that the way he rolled the die controlled individual outcomes.

Ben now went quiet again, until I asked him what he was thinking about. He commented that the axis of rolling the die did not explain the outcomes as 2 and 3 were not opposite to each other on the die.

Ben: Just seeing… if I was always rolling it in a way so it only lands on 6, 5 1, 4. But that wouldn't work, or make sense… it stays away from 2s and 3s, but it won't cos they're not next to each other – but they are.

Here again he was reasoning a local perspective, trying to find an explanation for the short sequence of outcomes observed. But in the next sentence he switched to expressing a global explanation.

Ben: It might be weighted more heavily on the 2 and the 3, on the inside, I was just thinking. If the weight's heavier there it will be less likely to turn that way.

Ben rolled a 1 and remarked that the die seemed more random now.

Ben: The more you do it, you know, the more different… But at the start it was all the same. So the more you do it the better the results you get, I suppose.

He found a possible global explanation and he tried to stabilise this idea in his mind. The next throw produced a 5.

Ben: …It should, unless it's weighted, be completely random. But at the start it just seemed to be 5s and 1s. But then it… just got a lot more mixed as it went down, so I suppose… it's just… more and more of a dice and, sort of less chance that the odd number will count for so much. You got a couple of 5s at the beginning, then, later as you go on, you'll get more of the other numbers as well. In theory, I think. …Although I haven't got that many 2s still.

As he moved towards a stable global perspective, Ben was holding in tension the two contrasting ideas of randomness (by which he means equiprobability), and bias – and he expressed them alternately. These were the apparently conflicting global interpretations for Ben: prior belief and a global frequentist view, possibly emerging from the aggregation of the observed outcomes.

As soon as he had expressed the idea of randomness, Ben reverted to discussing the bias. He rolled another 5 and reverted to a local perspective.

Ben: But I have got quite few 5s I think. …But that could just be the way I'm rolling it.

Over the next few throws, Ben's concern about bias diminished as he obtained more 2s and 3s. He remarked again on the apparent randomness.

Ben: Maybe it's just… I suppose it could just be a completely fair dice… It does have quite a few 5s, but that just might be me rolling it, rather than the dice would be weighted or something.

CONCLUSIONS

Ben's ideas were strongly affected by short run behaviour of the die. When the sequence of recent outcomes did not include one or two of the possible outcomes, he tried to explain the apparent bias. When the missing outcomes had appeared once or twice, he described the behaviour of the die as "random". Sometimes "random" was "the absence of pattern", and this cue was switched on and off by short-term changes in the sequence of outcomes.

Ben's interpretation of the outcomes was also influenced by the fact that he did not know how much variability to expect from a fair die. For example, he did not know how many throws he might need to wait until all six outcomes had appeared at least once, or how often should the most commonly occurring outcome appear in the first *n* throws of the die. Therefore Ben could not judge whether he had seen too many 5s, or whether the waiting times he observed before the first 3 and the first 2, were appropriate in a fair die. To refine his judgement of whether a die was fair, Ben needed intuitions about variability. Understanding of variability is important in reconciling the local and the global views of randomness.

In all the interviews, shifting attention between local and global perspectives was common, and the phenomenon seemed to be fuelled by a desire to draw conclusions from short sequences of outcomes. This in turn seems to be related to a poor understanding of variability.

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INTEGER INSTRUCTION: AN EXPERIMENTAL COMPARISON

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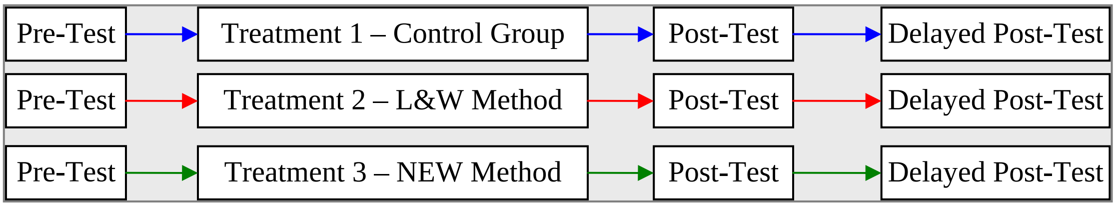
Two versions of Linchevski & Williams' 'dice games' method for integer addition and subtraction were experimentally contrasted. We describe the methods, present some statistical analyses and discuss the findings. We finally suggest more attention to situated meanings and intuitions in realistic contexts is needed.

INTRODUCTION

Linchevski & Williams (1999) in the *dice games* method for integer addition and subtraction targeted students' intuitive reification of integers, capitalizing on situated intuitions – a key to the intuitive instruction of integer addition and subtraction. As part of an ongoing PhD, a quasi-experimental design is examined where the original *dice games* method (Linchevski & Williams, 1999) is *L&W method* and a variation of it is *NEW method*. The two methods were contrasted with a control group (no integer addition and subtraction instruction), as shown in figure 1:

Figure 1: The quasi-experimental design



The students completed a pre-test, a post-test and a delayed post-test which (via Rasch analysis) produced three repeated measures for each student on an integer ability (IA) scale. We present the differences between L&W method and NEW method followed by some results of their statistical analyses based on the IA measures and discuss the consequences.

THE TWO EXPERIMENTAL METHODS (L&W AND NEW METHOD)

As L&W method (the original *dice games*) is detailed in Linchevski & Williams (1999) and in Koukkoufis & Williams (in press), here we stress the differences. Both contain 4 games that students play in groups of 4, divided in two teams: yellow and red team for L&W method; teams 1 and 2 for NEW method. In both methods, students throw in turns: in game 1, a yellow and a red die; in game 2, these and an *add/sub die* (i.e. a die giving *add* or *sub* (subtract)); in game 3, only an *integer die* (scoring -3, -2, -1, +1, +2, +3) replacing the red and yellow die; in game 4, the *integer die* and the *add/sub die*. Each group has two abacuses where their points are recorded. In L&W method, each abacus is used by a yellow team and a red team student recording points for both teams. In NEW method, each abacus is used only by one team for recording points for that team. Yet, the dice have different meanings in the methods.

In L&W method, the yellow and red die results are *points for the yellow* and *for the* *red team* respectively. The team that gets 8 points ahead of the opponent wins. Despite the teams only *getting* points (not losing any), as they try to get ahead, a point for the yellows is a point less for the reds. The students must understand that the dice scores can be cancelled (*cancellation strategy*) and that adding a point to one team is equal to taking a point from the other (*compensation strategy*) as these strategies are fair in the games. This *intuition of fairness* is crucial. Equally vital is in game 3 the transition from points *for the yellow* to *pluses* (i.e. *3 for the yellow* is *+3*) and from points *taken from* the yellow (or *points for* the red) to minuses (i.e. *3 for the red* is *-3*). Similarly, *+3* is *3 taken from the red*: positives and negatives have opposite meanings for the teams.

In NEW method, yellow and red points are *winning* and *losing points* respectively for both teams. Thus, the teams are *both winning and losing* points. To win, a team needs an overall score of 8 winning points (e.g. if a team has 2 losing and 10 winning points, its overall score is 8 winning points). Based on the game *intuition of fairness* (as in L&W method), but now also based on *winning and losing intuitions*, the students here too construct a *cancellation* and a *compensation strategy*. In game 3 winning points become *pluses* and losing points become *minuses* for *both* teams (e.g. +3 is 3 winning points): here integers have the same meanings for both teams.

We hypothesize that the different meanings of the yellow and red die and the additional winning and losing intuition in NEW method are at the heart of the methods' differences. First, negative integers in L&W method are not actually sub-zero quantities: they are the scores of the red team and therefore positives and negatives have the *same qualities* (they are above zero), but are *opposite amounts* because of the game context. We believe this can cause some confusion with integer comparison and reduce the effectiveness of L&W method. In contrast, in NEW method positives and negatives have opposite qualities, as negatives are below zero (they are *losing points*). We believe NEW method students will not experience the same problem. Secondly, in NEW method the students have the extra intuition of winning and losing to build new knowledge upon. As it is not just an intuition limited to the games context (winning and losing are opposite in almost all aspects of life), we hypothesize that it will allow more students to make integer operations intuitive. Further, we assume that if NEW method students are confused at some point (before or after instruction completion), it will be easier for them to reconstruct knowledge based on this game intuition than just on the intuition of fairness. Finally, because of the double meanings of integers in games 3 and 4 in L&W method, Linchevski and Williams (1999) report a counter-intuitive situation when negative integers *add* to the red team. We believe this counter intuitive moment will not exist in NEW method.

Due to these differences in the situated meanings of integers and in the situated intuitions, we hypothesize NEW method students will improve their integer abilities more, while L&W method students will improve their abilities also but to a lesser extent. In this paper we ask the research question:

RQ: What are the effects of the 2 experimental methods in students' attainment of integer conceptions?

ANALYSIS OF EXPERIMENTAL DATA THROUGH SAMPLE MEANS

In our experimental design year 5 students from one Greater Manchester school were examined, a class for each treatment. A summary of students' integer abilities by treatment and test type is presented in table 1.

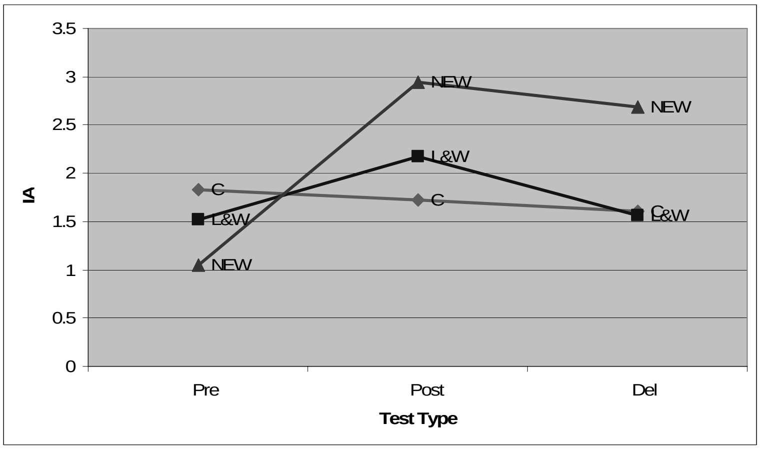
Table 1: Summary of students' IA measures by treatment and test type

	L&W Method			**NEW Method**			**Control Group**		
	Pre	**Post**	**Del**	**Pre**	**Post**	**Del**	**Pre**	**Post**	**Del**
Valid Measures	15	15	15	21	21	21	23	23	23
Total Cases	15	15	15	21	21	21	23	23	23

In the table we notice that the sample sizes for the three treatments by test type are quite small, reflecting limited access and the time-consuming design of the experiment (small group teachings; 4 one-hour meetings per group for teaching, testing and interviewing). This is a limitation for our experiment because some differences may not be found to be statistically significant. Yet, if they are found, we should not worry about the small sample sizes for these differences: there is a danger that we may not find a statistically significant difference, but there is not a danger that a statistical significance does not exist. Another limitation is that the small-group instructions allow increased attention to each child and favour intense learning. This limitation affects the comparison of L&W method and NEW method with the control group, but not the comparison between methods L&W and NEW.

Using the IA measures, the experimental samples were examined using descriptive statistics. Here we will only examine the sample means, represented in figure 2.

Figure 2: The sample means by treatment and test type



From figure 2, we see that in the pre-test, NEW method has the lowest mean, L&W method is 0.47 logits higher and the control group has the highest mean (higher of NEW method by 0.79 logits and of L&W method by 0.32 logits).

In the post-test, the control group mean remained almost unchanged (decrease of 0.11 logits – effect size (ES) = -0.10). In contrast, L&W method presented an increase of

0.66 logits (ES = 0.35). The change in NEW method was more impressive: it presented an increase of 1.9 logits (ES = 1.1). This great increase not only covered the pre-test difference between NEW method and the control group, but NEW method had a higher mean than the control group by 1.2 logits. Finally, though in the pre-test NEW method had a lower mean than L&W method, in the post-test NEW method mean was higher than L&W method mean by 0.77 logits. Thus, for the samples, NEW method outperformed both L&W method and the control group between the pre-test and the post-test, while L&W method also did better than the control group.

In the delayed post-test, again the mean of the control group remained almost unchanged (decrease of 0.12 logits – ES = -0.15) in comparison to the post-test, but this did not apply for the two experimental groups. L&W method presented a decrease of 0.61 logits (ES = -0.32). Further, though the delayed post-test mean of L&W method remained 0.05 logits higher than the corresponding pre-test mean (ES = 0.03), this difference is so small that one can hardly claim that eventually the acquired knowledge was retained. On the other hand, in NEW method the sample mean presented a smaller decrease of 0.26 logits (ES = -0.16). Thus, for NEW method the delayed post-test mean is higher than the corresponding pre-test score by 1.6 logits (ES = 1.3). As a result of this decrease, the gap between the NEW method and the control group became 1.1 logits. Finally, NEW method mean is higher than L&W method mean by 1.1 logits.

INVESTIGATION OF GENERALIZATIONS TO THE POPULATION

To examine the generalisations to the population regarding the three treatments, we have constructed (through the use of the software 'R') the following model:

Integer Ability (IA) = Treatment + TestType + Treatment X TestType + Subject

In this model: (a) Integer Ability (continuous variable) is the ability measured by the scale we have constructed through Rasch analysis; (b) Treatment (unordered categorical variable with 3 levels) refers to the control group, L&W method and NEW method; (c) TestType (ordered categorical variable with 3 levels) refers to the pre-test, the post-test and the delayed post-test; (d) Treatment X TestType is the interaction of the two above factors, which allows the comparison between the factors; and (e) Subject (unordered categorical variable) indicating the students participating in the experiment. The inclusion of the students in the model allows us to construct a more appropriate model for our repeated measures design. Regarding the use of regression models for repeated measures designs, see Hutcheson & Sofroniou (1999).

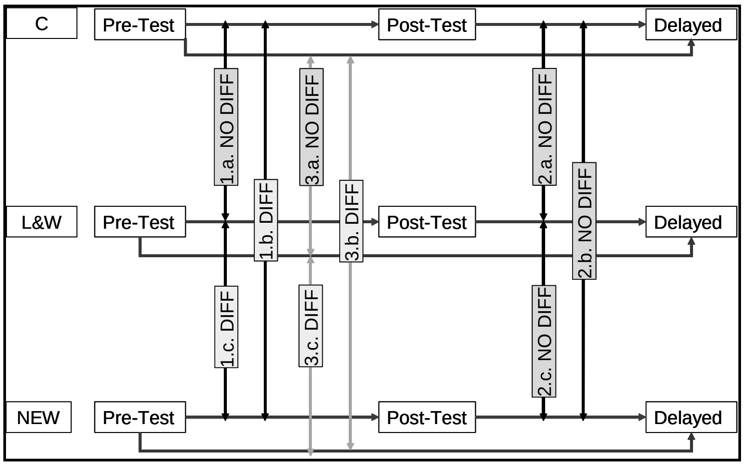
The produced model presented $R^2 = 0.66$, meaning that the model predicts about 66% of the variance of the dependent variable. As the F-statistic gave a p-value of 6.494e-09, we can say confidently that the linear relationship is significant. Based on these two characteristics of the model, we conclude that the model is appropriate and therefore can be used for statistical inferences about the population.

Based on the model, the null hypotheses that there were no statistically significant differences between the following differences were examined:

1. The difference of the means of integer ability (IA) in the pre-test and the post-test for: a. the control group and L&W method (CPost - CPre = L&WPost - L&WPre), b. the control group and NEW method (CPost - CPre = NEWPost - NEWPre), and c. L&W method and NEW method (L&WPost - L&WPre = NEWPost - NEWPre).
2. The difference of the means of IA in the post-test and the delayed post-test for: a. the control group and L&W method (CDel - CPost = L&WDel - L&WPost), b. the control group and NEW method (CDel - CPost = NEWDel - NEWPost), and c. L&W method and NEW method (L&WDel - L&WPost = NEWDel - NEWPost).
3. The difference of the means of IA in the pre-test and the delayed post-test for: a. the control group and L&W method (CDel - CPre = L&WDel - L&WPre), b. the control group and NEW method (CDel - CPre = NEWDel - NEWPre), and c. L&W method and NEW method (L&WDel - L&WPre = NEWDel - NEWPre).

The answers to these hypotheses were schematized with arrows in figure 3:

Figure 3: The comparisons for the hypotheses schematised



As 'DIFF' means that there is a statistically significant difference and 'NO DIFF' means that a not statistically significant difference was found:

1. In all the comparisons between the control group and L&W method no significant differences were found (hypotheses 1.a., 2.a. and 3.a.). However, this may be a small-sample effect. This is a limitation of our small sample sizes.
2. In the comparisons between the control group and NEW method, the differences in hypotheses 1.b. and 3.b. were statistically significant, but for hypothesis 2.b. it was not.
3. In the comparisons between L&W method and NEW method, the differences in hypotheses 1.c. and 3.c. were statistically significant. For hypothesis 2.c. the difference was not statistically significant. This may be a small-sample effect, but we cannot tell. Therefore, this is a limitation to our analysis.

DISCUSSION

We have presented the differences of two versions of the *dice games*, which we called L&W method and NEW method. Our analyses have shown that both methods improve students' integer abilities between the pre-test and the post-test, though the improvement for L&W method was not statistically significant – we suggest due to the small samples. Further, for both methods students' abilities decrease between post-test and delayed post-test, but for L&W method the decrease is so large that almost all the new knowledge seems to have been lost, whereas in NEW method most of the new knowledge is retained. In both cases the decreases were not statistically significant, but we suspect that for L&W method this is again a small-sample effect. Based on all these evidence, we can be confident that NEW method was more effective to L&W method and to the control group. Regarding L&W method, we cannot say for certain that it is preferable to the control group but we assume it should be, because in this method too the students produce intuitive strategies (Koukkoufis & Williams, 2006) and they have shown an increase of 0.66 logits (ES = 0.35) between the pre-test and the post-test which is noteworthy. To be sure, a comparison of L&W method with the control group with larger numbers of students would be needed.

Concluding, we believe that this experiment – in addition to the better learning outcomes produced and retained through NEW method – enforces the significance of situated meanings (e.g. the situated meanings of integers in our case) and situated intuitions (e.g. here, the intuitions of fairness and of winning and losing) in realistic contexts. Particularly, we suggest that the changes in these game characteristics in NEW method have led to a chain of differences in the games and explain these significant differences in their learning outcomes. We take advantage of this opportunity to call on the need for more attention on situated meanings and intuitions in realistic contexts.

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SUBTRACTION OF FRACTIONS THROUGH THE EYES AND EARS OF FIFTH GRADE MODELLERS

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The central question addressed in this paper concerns the ways in which modelling activities ground the conditions for a group of fifth graders to experience a progression in their awareness of subtraction of fractions. The teacher's narratives along with students' written work and transcripts of audio-taped class discussions constitute the primary data source for analysis. My attention is particularly drawn to a close examination of a teaching episode that appears to serve my research query. The study documents strong evidence that students could refine their fractional reasoning when exposed to a learning environment that sensitises them to detect problematicity through confronting impasses publicly, questioning themselves and peers, conjecturing and welcoming broken expectation.

INTRODUCTION

The Rational Number Project (Behr et al., 1992, 1993; Lesh et al., 1987) as well as Kieren (1988, 1993) marked a notable line of research in the area of teaching and learning fractions. Their major premise was rooted in splitting the concept of rational number into subconstructs and then binding up the complementary meanings arising from each system (e.g., part-whole, operator, etc) to achieve an integrative understanding of fractions. Though I acknowledge the importance of content in construing the meaning of such a complicated concept I remain dubious about the effectiveness of concentrating learners' focus on the particularities of each subconstruct; my uncertainty stems from Mason's (2001) caution that 'if the centre of gravity of students' attention is on the particularities of the example or model, then their awareness of the process is likely to be at best implicit and indirect' (p48). Against this background, the current paper describes an attempt to enter the world of a group of fifth graders (mostly by listening) and examine how in the midst of spoken and unspoken utterances students negotiate the meaning underlying subtraction of fractions.

THEORETICAL BACKGROUND

The study evolves around two spiralled frameworks: 'the constructivist teaching experiment' (Cobb & Steffe, 1983) and Mason's (2001) perspective of mathematical modelling. In the constructivist view, opportunities for learners to build their knowledge arise as they interact with both the teacher and their classmates; mathematical awareness is, thus, not arbitrarily generated but, instead, 'constrained by an obligation to develop interpretations that fit with those of other members of the classroom community' (p137). Sensitizing students' attention to the various stimuli conveyed either by peers or the teacher is an essential component of modelling, as well. For Mason (2001) the heart of modelling lies in an interactive shift between experiences. Once formulated, a model directs modeller's thinking 'through its particular stressings and ignorings' (p50). The key role of teachers in both the constructivist and the modelling perspective is to establish a classroom culture in which learners volunteer to struggle publicly, welcome the unexpected, feel free to conjecture and filter their world through a mathematical lens.

METHODOLOGY

The current work portrays part of a classroom teaching experiment implemented in an elementary school in Cyprus (September 2005 to June 2006). The participants in my study were a group of 22 11-year-olds (10 boys and 12 girls) taught by the author. Arising from my twofold profile as a teacher-researcher was the commitment to address all the objectives of fifth grade mathematics set by the national curriculum. Therefore, I coordinated only once a week modelling activities around fractions. In the end of each lesson I translated my experience from Greek to English and reported it in a journal; the teacher's narratives along with students' written work and transcripts of audio-taped class discussions constitute the data for analysis. As initially conceptualised, this paper aims to transmit a sense of how it is like to learn to subtract fractions in a public school classroom in which knowledge is not lectured but negotiated. The following research question is submitted as an entry point: How does the use of modelling activities in teaching subtraction of fractions influence students' learning processes?

FINDINGS

I will hereby display a teaching episode that emerged in March 2006. That day children struggled publicly to model the subtraction fact 1-2/3. It is important to note here that in the previous months my fifth graders have not been exposed to any formal instruction of fractional algorithmic procedures but have, instead, actively construed their own fractional knowledge 'in the midst of manipulating and expressing' (Mason, 1987, p210); rewording Mason's definitions to fit in the current context, I describe manipulating as the act of drawing either in one's head or on a palpable surface (paper or board) rectangular models of fractions. Expressing, on the other hand, points to learners' attempts to voice their inner experience either to themselves or others. Along with students' paper-and-pencil manipulations I occasionally employed technology-based representations as a window through which my students could glimpse a more accurate version of their own constructed models.

1	Teacher:	Let's suppose that we have 1 whole minus 2/3

[Ria is coming on the board and developing Figure 1 (lines 1-7)]

2	Ria:	I am drawing an area model and I leave it as a whole and I make another one and I divide it in two equal parts…. I divide it in two equal parts horizontally and in three equal parts vertically.
3	Teacher:	Yes?
4	Ria:	[Pause]
5	Teacher:	Think aloud Ria, it doesn't matter if you make any mistake.

6	Ria:	Then I will do the same with this one.
7	Teacher:	Ria did the same drawing in the first area model as in the second one.

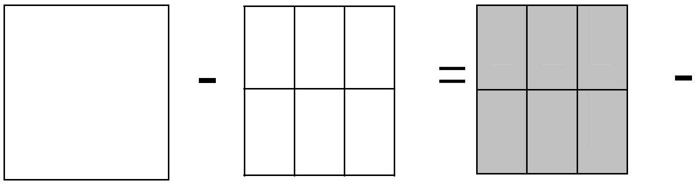

Figure 1: Ria's first attempt to model 1-2/3

8	Ria:	No sir I realized my mistake; I should have divided it into 3 equal parts and take the two [Ria is developing Figure 2: lines 8-22].
9	Teacher:	Okay, try this way then. So you divide the second one into 3 equal parts and you take the two. How about your first area model? What do you have there?
10	Ria:	The denominator is 6 in both cases…
11	Teacher:	Ria did you say that the denominator is 6 in the second one, as well? Could you explain a bit more?
12	Ria:	No sir it is not 6.
13	Teacher:	What is it then?
14	Ria:	I divide the second area model into 6 small squares…. Hm, I divided it in half so that I could get six squares.
15	Teacher:	And how much you have shaded now?
16	Ria:	This one will be 6/6 minus 4/6.
17	Teacher:	Why is it 4/6?
18	Ria:	Because I divided this square in half and then I did…
19	Teacher:	Correct! Could you tell us Ria how much did you shade now?
20	Ria:	4/6.
21	Teacher:	Well done Ria and then what will you do?
22	Ria:	6/6 minus 4/6 is 2/6.

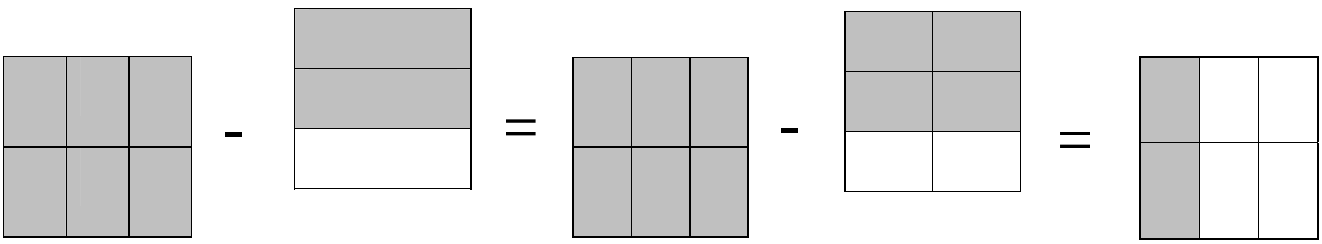

Figure 2: Ria's second attempt to model 1-2/3

23	Teacher:	I saw some kids raising their hands. Who would like to say?
24	Jim:	Sir I disagree with Ria's way.
25	Teacher:	Would you like to say a bit more?

26 Jim: You said that the second area model is 2/3. We could divide the one whole into 3 parts and shade all of them and then the 2/3 there to transfer the columns…

27 Teacher: How about coming on the board Jim and show us? Jim will now show us another way of subtracting 1-2/3.

[Jim is coming on the board and developing Figure 3 (lines 28- 32)]

28 Jim: I draw an area model and I divide it into 3 columns and I shade 1…

29 Teacher: Why one?

30 Jim: I mean 2 because we have 2/3 and then I draw a whole. I will divide it into 3 horizontally and color all the rows. Then I will exchange the columns and we will have this one 9/9 and this one I will transfer the 3 horizontal. I will get here 6/9.

31 Teacher: So what would you have?

32 Jim: 9/9 minus 6/9 equals 3/9.

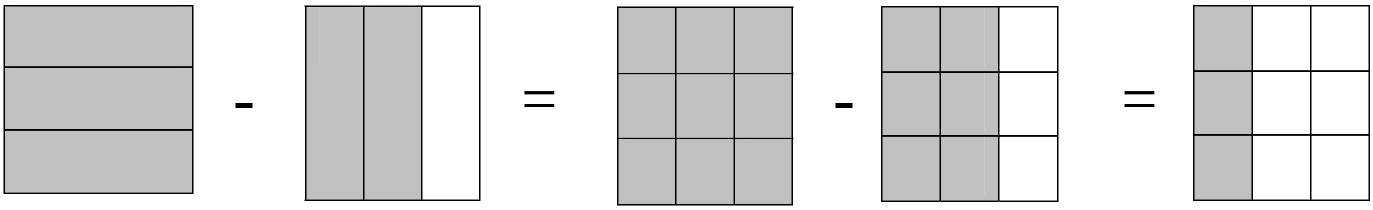

Figure 3: Jim's model of 1-2/3

33 Larkos: Is this correct sir? I think it's not correct…

34 Teacher: What would you like to say Larkos?

35 Larkos: I have another way. I will take one whole and leave it as a whole and then I will draw the 2/3…three rows and I will transfer the three rows into the one whole.

36 Teacher: Well done, and?

37 Larkos: Then because it is one whole I will color it all and then the 2/3 I will make it minus… 3/3 minus 2/3 and I will get 1/3.

38 Teacher: So we've seen three different ways of subtracting 1 – 2/3. All of the ways are correct. Let's now see how the computer works out this example.

39 Ria: It moved the one whole into the 2/3 and the 2/3 into the one whole?

40 Teacher: And why it did so?

41 Ria: To facilitate us to get the same denominators.

42 Teacher: What it did here?

43 Ria: It made them all thirds and now it will subtract… and we will get 1/3.

44 Teacher: What denominators did you use previously Ria?

45 Ria: Sixths.

46 Teacher: Sixths, it doesn't matter. And Jim?

47 Jim: Ninths.

48 Syria: But if we simplify 2/6 that Ria found won't it be 1/3?

49 Teacher: Great Syria! All the three ways are correct. The important thing is to understand the way.

50 Xenios: Can't we make the denominator 12?

51 Teacher: Yes.

52 Xenios: Basically the denominators should be multiples of 3.

DISCUSSION

There seems to be much "prima facie" evidence in the transcript signifying that students' fractional reasoning flowed as an extension of a growing awareness of an innate experience coupled with an externalisation of that experience (either in words or drawings). In the midst of Ria's public struggle to model 1-2/3 we are made witnesses of an impasse (lines 1-7); the rectangular models she developed at first do not seem convincing to her and this instantly brings to the fore a broken expectation. Instead of suffering in silence (line 4) the girl activates promptly an alternative way (line 8) whose initial vagueness (lines 10-12) motivates her to successively refine it. This finding reveals the great learning potential arisen from offering students the flexibility to discern problematicity within their own constructed models. Learning to subtract fractions is ill-defined if is interpreted in terms of input (change-into-common-denominators formula) and output (correct answer); this practice though efficient most of the times does not serve the very essence of teaching, that is, to broaden our students' horizons so that they could draw on their own conceptual repertoire when they become confused (like Ria for instance) or cannot recall a memorized rule.

Jim's (lines 23-32) and later on Larkos' (lines 33-37) publicly shared "opposition" to Ria's and Jim's approach, respectively, along with the implementation of technology-based representations sparked productive stimuli which eventually formed the ground for the transition Xenios exhibited from apparatus-situated cognition to abstract generality (line 52). In essence, Xenios re-invented the premise that to subtract a fraction from an integer the common denominator of the two numbers must be a multiple of the denominator of the subtrahend fraction. A question-mark reasonably generated is how this boy in the midst of a long silence formulated such an assertion. Instead of submitting my own interpretation I step back and consider the alternative; what if I assumed the role of the expert and displayed from the very beginning a series of routinized steps suitable for each distinct case of fractional subtraction? Would my students discern problematicity at all? Apparently not for, commonsensically thinking, there is no benefit earned from questioning the "authority". Hence, this line of research claims that true fractional reasoning emerges when students are channelled to sensitize their eyes and ears so that they could stress, as Syria did for instance (line 48), 'certain features of the what they see [and hear] as a result of their expertise' (Mason, 2001, p56).

I have hereby attempted to exemplify one perspective of what it is like to be a learner of subtraction of fractions. I offered my own situated experience as a public school teacher as evidence that incidents of conjecturing, surprise and questioning along with the use of modelling activities (technology-based or paper-and-pencil) give rise to multiple transitions in fifth graders' thinking. For the knowledgeable other, this study might not fit certain standards of methodological order, rigor and objectivity. To this end I resort to Mason's (2005) reflection; 'what has happened to the emergent, to the intuitive sensitivity that can develop between teacher and learner and that occasions the deep learning that we all value and seek?' (p472)

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DEVELOPING ON-LINE QUESTIONNAIRES FOR MATHEMATICIANS

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In this paper I will discuss the development of an on-line questionnaire that I have designed for my dissertation research. By employing this questionnaire, I aim to gauge university mathematicians' use of Computer Algebra Systems (CAS) in undergraduate mathematics courses and to understand their thinking about the advantages and disadvantages of CAS use in university-level teaching. The development of the questionnaire is based on an interview study with mathematicians that I conducted in 2005. Thus, I integrate issues that emerged from this earlier study with concerns described in the mathematics education literature. The complexity of the questionnaire design is complicated by the fact that I am examining mathematicians in three countries, Hungary, the United Kingdom, and the United States, which requires me to consider different aspects of international comparative research and differences in cultures. I will report on the difficulties that I encountered during the design process of this questionnaire and highlight issues which researchers must pay attention to when they decide to use on-line questionnaires.

INTRODUCTION AND AIMS

Anecdotal evidence suggests that Computer Algebra Systems are becoming an integral part of university-level mathematics teaching and learning. In the past two decades, a number of studies examined various uses of CAS in classroom settings and students' learning in CAS-equipped environments in higher education. However, little attention has been paid to why and how CAS is being integrated into the university curriculum, what factors influence CAS integration, or to its sustained use in a university environment. In contrast, several school-level studies examined the integration of technology and its sustained use in schools (Hennessy, Ruthven, & Brindley, 2005). In addition, from time to time large scale studies and international comparative surveys mapped the use of technology in schools (Becker, 2000; Gonzales et al., 2004). These school-level studies demonstrate that technology is still lightly used in schools despite the heavy investment by schools and governments and that technology integration greatly depends on teachers' conceptions of technology and social/cultural factors. In my study, I aim to investigate the extent of current use of CAS, mathematicians' conceptions of CAS-assisted teaching and the influence of social and cultural factors on technology use at the university level.

DESCRIPTION OF THE RESEARCH PROJECT

For my study I chose a particular software called Computer Algebra Systems because CAS is the most widely used mathematical software in university-level mathematics. In addition, I wanted to choose a technology application that is not only a general software but is directly related to mathematics. Building on the research conducted in schools, I posed three questions to explore CAS use and CAS integration at universities:

1. To what extent and manner are Computer Algebra Systems currently used in university mathematics departments?
2. What mathematical and pedagogic beliefs and conceptions do mathematicians hold with regard to CAS including factors influencing their professional use of CAS?
3. To what extent do nationally situated teaching traditions, frequently based on unarticulated assumptions, influence mathematicians' conceptions of and motivation for using CAS?

The first question attempts to provide an overview of the current use of CAS in universities similarly to quantitative studies conducted at the school-level. The second question examines mathematicians' conceptions revealed by school-level studies as a key factor of technology integration into the mathematics curriculum. Finally, the third question investigates the influence of teaching traditions on CAS integration. By answering these questions I aim to provide a basis for researchers to build research projects on in order to more closely investigate issues of technology integration. Furthermore, I hope that I will be able to highlight differences and similarities between findings on the use of technology at universities and at schools.

The posed questions obliged me to employ conflicting research paradigms. The *Mixed Methods* approach, backed by pragmatist philosophy as Johnson and Onwuegbuzie (2004) argue offer a plausible resolution of this conflict. Therefore, in this study, I utilize an *across-stage mixed-model research design* (Johnson & Onwuegbuzie, 2004). In accord with this design, during the past year, I conducted a qualitative study to uncover issues that can be further investigated in a quantitative investigation (phase I). I conducted exploratory interviews with 22 mathematicians at a range of universities in Hungary, the United Kingdom, and the United States. In addition, I observed classes and collected course material during my university visits. Based on the results of this investigation I have developed a quantitative study (phase II) to further examine issues which arose in the first phase.

Results of the first phase of the study support school-level findings and show that the integration of CAS into university mathematics curricula is heavily dependent on mathematicians' conceptions of CAS and CAS-assisted teaching (Lavicza, in press). In addition, the study revealed a number of personal and external factors that influence CAS integration and sustained use of CAS in university mathematics education [1].

ADVANTAGES AND LIMITATIONS OF ON-LINE QUESTIONNAIRES

The organization of emerged issues – seeking connections

The first phase of the study identified a large number of issues that would be worth further investigation but the restricted length of the questionnaire forced me to identify and investigate the most significant issues. Over a 7-month period, I worked on the selection of these issues and piloted questionnaire items in several rounds. Finally, I developed a questionnaire that supplies data for the following clusters of variables:

- Mathematicians' personal characteristics/institutional backgrounds
- Mathematicians' current use of CAS in teaching
- Mathematicians' conceptions:
  - CAS viability in mathematics education
  - CAS self-efficacy
  - CAS Role in Mathematics Literacy
  - CAS-assisted Teaching and Learning – affordances and dilemmas

In the analysis of the data I aim to establish connections among the three clusters of variables (Figure 1).

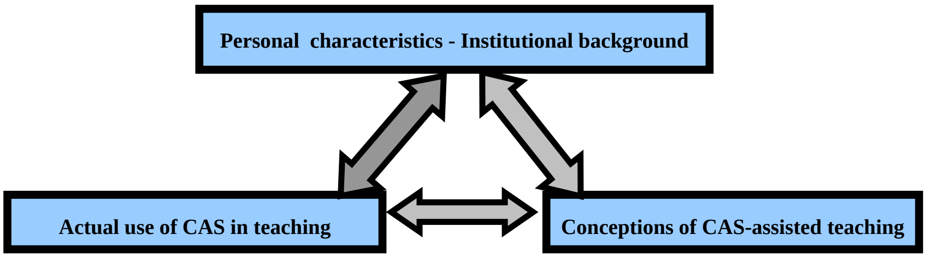

Figure 1

The 'personal characteristics and institutional background' clusters describe mathematicians' background and their work/career history. Essentially the independent variables of the study are situated in this cluster. The "actual use of CAS in teaching" cluster attempts to reveal the extent of CAS use in university-level teaching and learning. The "conceptions of CAS-assisted teaching" cluster attempts to expose mathematicians' thinking about CAS-assisted teaching. As Figure 1 shows these connections, I will examine how mathematicians with particular personal characteristics and institutional background think about CAS-assisted teaching, or how/why they use CAS in their teaching. In this way, I might be able to identify characteristics of particular mathematicians that make them likely or unlikely users of CAS-assisted teaching. Identifying such characteristics can be valuable for the development of CAS training programmes. I also seek to explore relations between conceptions of mathematicians and their actual use or non-use of CAS. This information can also result in valuable characterization of mathematicians. In addition to the three clusters, I aim to expose the reasons why mathematicians begin or avoid using CAS in their teaching practice.

International considerations

Due to the international characteristic of my study I had to consider the differences among cultures while developing my questionnaire. Osborn (2004) offers a suitable framework by establishing four equivalence criteria:

1. Conceptual equivalence – examines if concepts used in the study have any equivalent meaning in different cultures.
2. Equivalence of measurement – develops equivalent indicators for concepts.
3. Linguistic equivalence – suits the meaning of the text of the questionnaires to the particular cultures.
4. Sampling equivalence – ensures the representativeness of the sample in a particular country or culture.

On-line questionnaires

In order to develop a comprehensive study, I have to reach a sizable number of mathematicians. Due to the financial and time restraints, I decided to utilize on-line survey techniques to conduct the second phase of my study. Schonlau, Fricker, and Elliott (2001) list a number of advantages of web surveys over surveys administered on paper. On-line surveys are usually: 1) Less time consuming; 2) just as good or better than traditional surveys; 3) much cheaper to administer; 4) easier to execute. However, researchers also warn about the potential drawbacks of web-based questionnaires as 1) various technological problems might arise 2) responders may have different computer expertise causing loss of data quality 3) it is difficult to ensure data security 4) it is hard to draw a random sample 5) unlike paper questionnaires, responders are not in control of the entire questionnaire 6) response rate of web-surveys are usually lower than paper/mail surveys. For my study advantages of on-line questionnaires outweigh the disadvantages because examining mathematicians as the population of the study provides plausible resolutions for difficulties that generally arise in on-line questionnaire studies.

Sampling issues

One of the most daunting problems arose in my study when I began developing the sampling strategy that satisfies Osborn's (2004) sampling equivalence criteria. While formulating my sampling frame I decided to develop sampling that is the most representative for the selected country rather than trying to design a strategy that is applicable for the three selected countries.

The large number and great variety of higher education institutions in the US (about 4000), compared to the UK and Hungary, caused a considerable difficulty in developing the sampling frame. Finally, I was able to restricted my sample to 1478 US, 157 UK, and 52 Hungarian institutions. After estimating the population of mathematicians (35,000 in the 3 countries), I selected 3,500 mathematicians following a rigorous sampling strategy.

Issues of response rate

Besides developing an appropriate sampling frame, acquiring an acceptable response rate is the most crucial issue that contributes to the validity of the study. Most on-line and traditional survey methodology papers discuss response rate as a key issue (Cook, Heath, & Thompson, 2000; Couper, Traugott, & Lamias, 2001; Manfreda, Batagelj, & Vehovar, 2002). In spite of attempts to increase response rates with a variety of techniques, Dillman, Tortora, and Bowker. (2001) suggest that response rates for all kinds of surveys have been declining since the early 1990's. This tendency especially accelerated due to the emergence of on-line questionnaires. People are receiving an increased number of solicitations to participate in research studies or marketing research, and they are unlikely to respond to most of these appeals. In addition, technology contributes to lowering response rate because it is fairly easy to delete e-mail solicitations from electronic mailboxes. However, the on-line survey literature offers several ideas for the enhancement of higher response rates. Authors suggest that it is important to distinguish the study from other research projects, and that researchers should make potential participants interested (Sax, Gilmartin, & Bryant, 2003). In addition, it is important to properly invite participants, act upon their requests/feedback, and remind them to fill in the questionnaires. Moreover, keeping the questionnaire reasonably short - a maximum of 20 minutes - will increase participation (Solomon, 2001). Finally, offering incentives may prove to be an effective tool to increase response rates. I am using these considerations to improve the response rate of my study.

Visual appearance

Because web-based questionnaires offer a wide range of opportunities for different visual design, this topic is extensively discussed in the research literature. Studies suggest that simple and low-graphics design has highest success rates (Couper et al., 2001).

SUMMARY

I hope that my study will be able to provide a measure to show how CAS is currently used in universities in three countries. In addition, I hope that I will be able to offer insight how mathematicians envision the use of CAS in university-level mathematics teaching and learning. Certainly, this phase of the study may not offer deep insight into the details of CAS use, however, by identifying mathematicians and institutions, details can be further investigated in the continuation of this study. In addition, using an on-line questionnaire methodology, I will be able to contribute to the methodological debates in this area.

NOTES

1. Detailed results will be found in Lavicza (in preparation).

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THE COMPANY OF WORDS: USING CONCORDANCES TO DEVELOP LANGUAGE IN THE MATHS CLASSROOM

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THE PROBLEM WITH LANGUAGE IN MATHS CLASSES

Some thirty years ago, the linguist Michael Haliday argued that:

> The core of the difficulty in the mathematics classroom is that the teacher often understands and takes for granted the whole register of mathematics, and thinks only of the mathematical aspects of these items, whereas for the learner they may also be unfamiliar language - they are 'peculiar' English. It is therefore desirable that the mathematics teacher should be aware of the register of mathematics as a sub-set of English … To this end, mathematics educators and … English Language teachers should collaborate in the production of guidelines, illustrative descriptions and teaching materials concerned with this problem. (Halliday, 1975: 121)

Whilst numerous researchers have tackled the implications of this and sought to explain the 'peculiarities' of maths texts (Pimm, Morgan, Sfard, Barwell, Leung, Street, to name but a few) it remains the case that responses have largely been at a very local level with individual teachers or teams of teachers seeking to address the issue in their particular setting rather than there having been a more systematic approach to curriculum design informed by linguistic analysis. In this paper I will try to outline one possible way forward using a *corpus* of mathematical materials (taken from the SMILE scheme), interrogated using *concordancing* software, more on both of which later, but first I would like to give an illustration of how we can take things for granted about language when we rely on our intuitions. This based on a study by Tversky and Kahneman (1973).

Testing your intuition about language

In a typical sample of text in English, how many more times likely do you think it is that words begin with the letter 'k' than have 'k' as the third letter in them?

When I put this question to a group of mathematics researchers about two thirds of those of them intuited that there would be roughly twice as many words beginning with the letter 'k'. Interestingly, this was roughly the same as found in the Tversky and Kahneman (1973) study where 105 out of 152 people thought there would be more words beginning with 'k'. There are, in fact, likely to be about half as many. This is because we are far more likely to think of the first letter of a word than the third, but once you begin to think about, (bake, cake, fake, hake, lake, make, naked, rake, sake, take, wake…) it becomes clear that the initial assumption is wrong. Once you begin to 'problematize' mathematical English then the peculiarity Halliday identified becomes more obvious.

Some examples

The most obvious level is that of vocabulary. Few maths teachers would be surprised if when a child was asked for the first time what the difference is between 7 and 12 they were given a reply along the lines that one is odd and the other is even rather than the mathematically more expected answer ‘5’. Similarly with ‘similar’. These oddities continue at other levels.

Syntax: we say ‘three quarters ***is*** bigger than five eighths’ rather than ‘are bigger’ because fractions are treated as single entities so in spite of quarters being plural we use the singular form of the verb. Teachers of English as an additional language will have to explain to students whom they have diligently reminded of the need for subject-verb agreement in English why in maths it’s different.

People tend to assume that maths represents a pared down, objective form of language but it is also riddled with metaphor that can lead to confusion. A classic example is ‘multiplication makes numbers bigger’ and its concomitant, ‘division makes things smaller’.

WHAT NEEDS TO BE DONE?

It is clearly desirable, as Halliday said, that maths teacher should know about the register of mathematics if they are going to be able to guide their students through it (and, of course, there are particular challenges facing students learning English as an additional language through maths and *vice versa*. In the remainder of this paper I will try to describe how the use the use of a corpus and concordancing software might help with ‘the production of guidelines, illustrative descriptions and teaching materials’ that Halliday called for.

What is a corpus?

A corpus is simply a body of *authentic language* data. We now tend to think of it as being stored electronically and available for interrogation, e.g. through the use of a concordancer.

What is a concordancer?

A concordancer is a piece of software that enables its user to track and display patterns in a body of language by, for example, displaying key words in context (KWiC). Here is an example of a concordance of the word ‘poor’ in ‘A Tale of Two Cities’ book 1:

947	Miss, if the	**poor** lady had suffered so intensely
1884	the love of my	**poor** mother hid his torture from me
1615	stockings, and all his	**poor** tatters of clothes, had, in a long
1577	faded away into a	**poor** weak stain. So sunken and
1001	on your way to the	**poor** wronged gentleman, and, with a
1036	detachment from the	**poor** young lady, by laying a brawny hand

The word under investigation is displayed in bold in the centre with a defined number of words displayed on either side of it to allow the context to be clarified (most concordancers will take the reader to the original text when the word is clicked on so that the full context can be examined). This method of display is useful as it allows the investigator to identify real patterns of use as opposed to assumed ones. For example, if asked what the word 'poor' means, there is a distinct possibility that a first definition would be 'not wealthy', whereas the way it is used here emphasizes a secondary meaning, 'eliciting sympathy'.

Another way of displaying text in a concordancer is by using a frequency table, as in this display of 'feminine' adjectives used in a corpus of 19th century American fiction.

Adjective	**Women**	**Men**
little	112	59
dear	20	15
happy	15	9
pretty	14	9
sweet	13	8
lovely	12	3
pale	11	5
beautiful	9	6
eager	7	1
charming	5	8
delicate	3	7

This sort of display allows the reader to look at distinguishing patterns of use (here gender orientation) or simply to compare frequencies of different words in a text.

WHAT HAS THIS GOT TO DO WITH MATHS?

Concordancers have been used to analyse all manner of texts, literature, journalism, legal proceedings, school history, written and spoken, but I am unaware of much work on school mathematics. As a result, we have gained insights into how such texts are structured and used to create meanings. My ambition is to develop a corpus of school maths materials that will give us some insights into the language of mathematics and how students might be enabled to become more effective users of both. The data are drawn from some 1318 activities (including worksheets, workcards, posters, microprograms, games) of the total 1418 that were available at the time of compilation. This accounted for some 93% of the scheme, the remaining 7% being either unavailable to or held in an unsuitable format (i.e. computer activities without any written support or activities that involved no written text). The corpus itself is approximately 250,000 words. It seems reasonable to argue that, as the corpus included some 93% of the materials in the SMILE scheme and covers all levels of the National Curriculum, it serves as a basis for the analysis of the mathematics register as developed through this particular scheme.

In the following I will give an example of the sorts of things it might be possible to do.

What's a diagonal?

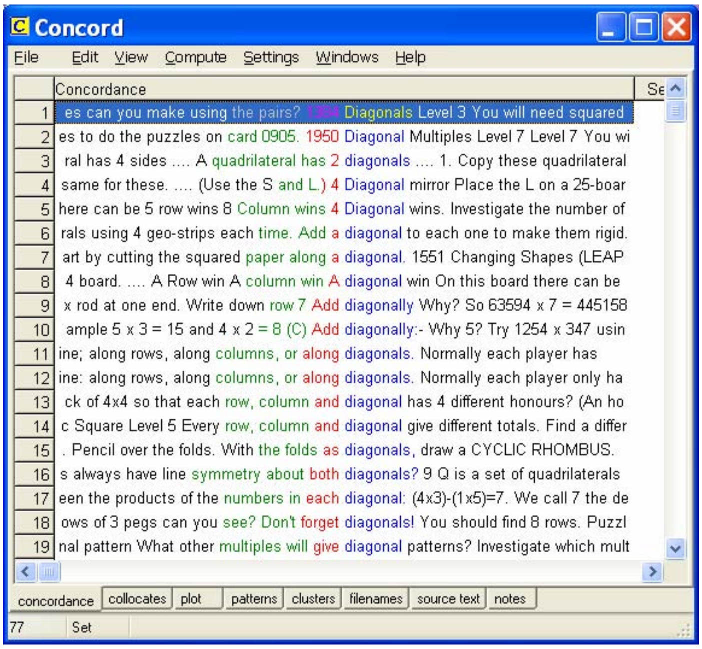
According to one mathematical dictionary:

> A diagonal is a straight line drawn from one vertex of a polygon (or solid figure) to another vertex. (Abdelnoor, 'A Mathematical Dictionary')

According to an ordinary dictionary

> A diagonal line goes in a slanting direction away from another line (COBUILD English dictionary)

I wanted to discover how the word was used in the materials the students were exposed to and the ways in which the two distinct meanings above were reflected in them. Analysis of the word (or in linguistic parlance 'lemma' which includes its multiple forms 'diagonal', 'diagonals', and 'diagonally' revealed that it appeared in as an adjective, 'diagonal moves are not allowed'; as a noun (singular and plural), 'The vector OR is the diagonal of a box…'; and as an adverb, 'You may not hop diagonally'. This demonstrated that the word was used in the scheme both in its mathematical meaning (as an attribute) and in its ordinary meaning (indicating orientation).



By looking at the frequency of occurrence I was able to discover some other interesting features. The corpus was structured in such a way that I was able to track what NC levels the word appeared at and so track its development through the materials. This threw up some interesting issues. For example, the word did not appear at NC level 2 in the scheme, suggesting that there might be a need or opportunity to plan in another activity. Further, there were particular clusters where it seemed to feature more heavily than at others (i.e. between levels 4 to 7), which may reflect its proper place within the hierarchy of mathematical concepts. What was most interesting, however, was that the lemma occurred 66 times throughout the scheme and of these 32 were in its technical, mathematical sense and 34 were in its non-mathematical sense. This ran counter to my intuition that a mathematics scheme would privilege mathematical meanings over ordinary meanings. That it doesn't has serious potentially implications for how students experience and understand mathematical terminology since the most common form they encounter is likely to have the most salience.

Students' understanding of mathematical terms

Here is an extract from an exchange between a group of Year 8 students. They were working on an activity that involved them identifying a shape from a set of attribute cards.

> **Jack:** [reading attribute card] 'At least one reflex angle'. No one knows what that is so you can't stop me.
>
> **Lucy:** No, wait. What is a reflex angle?
>
> **Jack:** I don't know. If you bop it then it'll kick you up.
>
> **Lucy:** No, wait. What's a reflex angle?
>
> **Jack:** It's when someone taps your knee and your leg jumps up.
>
> **Nadia:** Shall I go and ask the librarian?
>
> **Lucy:** One obtuse angle. What's obtuse? Nadia, while you're there, ask him what obtuse means.

However groan-worthy Jack's comment is (and clearly he loved it enough to use it twice and clearly Lucy was so used to his sense of humour that she completely ignored him) it does illustrate that if in doubt students are likely to fall back on a meaning that they know and where words in the mathematical register have a specialist meaning that is different from their meaning in ordinary English then problems can arise. When I interrogated the corpus to find out how 'reflex' was used I discovered that it made its first appearance at level 6, and none of the students were currently working at that level. So again, the concordance was useful in identifying a gap in the resources.

When Nadia returned from the librarian, the following exchange took place:

Nadia: [reading a written note she'd made] 'A reflex angle is an angle that is more than 180º. An obtuse angle is greater than 90º and less than180º.'

Jack: So it's the opposite of a right angle. It's an isosceles trapezium.

Mark: They're not parallel because that one's longer than that.

Jack: So?

Mark's final comment was also very revealing about his understanding of attributes; it would seem from this that for him parallel lines have to be equal in length. The corpus proved helpful here too. I had annotated all activities that had illustrations on them and so was able to go back and look at the salient features of these illustrations. A good example of this was the representation of rectangles There were 32 activities containing 48 images of rectangles. The modal ratio of the sides was 1:2, the mean ratio was 1:2.19. Roughly two thirds of the images depict the rectangle with the width greater than the height and all but a handful are oriented perpendicular to the page. This might account for comments such as these, taken from students who were asked to define the difference between a rectangle and a square:

Marcus: A rectangle is a four sided shape but it is longer

Celina: The lenth of square is more short then the rectangle

Andrew: A rectangle has two sides (the bottom and top) wich are wider than the two sides

Again, it would seem reasonable to argue that students' understandings of mathematical attributes are at least partly constructed by their exposure to images or the language used in the materials they are exposed to.

The challenge is whether we can develop and use corpora such as this one to identify gaps and opportunities for fresh interventions that teach our students what we want them to learn in a more linguistically (and therefore mathematically?) more principled way. A further ambition, with the needs of students learning mathematics through English as an additional language in mind, is to develop a language curriculum through maths. This would be to seriously address Halliday's challenge though whilst it is necessary it may not be sufficient. To address broader aspects of the child's learning experience, for example, the analysis would need to be extended to include affective aspects of the curriculum such as how it reflects cultural diversity – how many non-English names there are, what jobs people have compared to the mathematics they use, how disability is reflected in the materials and so on. It is through close examination of the particular that we may finally illumine the peculiar.

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IDENTITY AND UNDERGRADUATE MATHEMATICS: A DISCUSSION OF BARON-COHEN'S SYSTEMISER/EMPATHISER DICHOTOMY WITH REGARD TO MATHEMATICS UNDERGRADUATES AND ASSOCIATED GENDER ISSUES

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Continuing previous work on mathematics undergraduates' identities, we present and critique some of Simon Baron-Cohen's ideas on 'systemizing' and 'empathizing' and how these notions relate to gender. We are concerned that his association of systemizing with 'male brains' could be detrimental to female participation in mathematics.

INTRODUCTION

Mathematics education is influenced by many social and cultural forces and mathematics educators are wise to be alert to emerging ideas outside maths education. This paper focuses on one such emergent idea as expressed in the recent book 'The essential difference: the truth about the male and female brain' (Baron-Cohen 2003). This popularised book offers theorisation on two particular ways of thinking, 'systemizing' and 'empathizing', and claims that "the female brain is predominantly hard wired for empathy and the male brain is predominantly hard wired for understanding and building systems" (*ibid.* p1). This claim is a challenge to inclusive mathematics education as it distances those who are female-identified with the pragmatically systemizing process intrinsic to mathematics. Furthermore, the rhetoric of the book associates systemizing with autism and autism with deficiencies in empathy, thus by association positions systemisers as lacking in empathy. Given that our professional field is that of mathematics education and includes long-held interests in gender and mathematics, it behoves us to understand Baron-Cohen's claims particularly in the context of women and mathematics – and this includes females' identity, participation and agency with regard to mathematics.

THE 'ESSENTIAL DIFFERENCE'?

The central claim of the book, published for a wide audience, is stated as men and women have fundamentally different brains on average (*ibid.* p2). The book is written in a 'folksy' style and Baron-Cohen is quick to say that his topic is a delicate one and is not intended to be grist for reactionary gender oppression. The outline of the book's argument is that: every person has 'empathizing skills' and 'systemizing skills'. These skills are assessed by psychological tests that Baron-Cohen and colleagues have developed that assign to an individual a systemizing quotient', SQ, and an 'empathizing quotient', EQ. Results indicate males on average systemize and females empathize (*ibid.* p62) and thus the 'difference' of his title is key to mind and gender.

Roots of these concepts

The author is an academic psychologist whose field is the study of autism and this popular work draws on his research. Autism is defined is a 'triad' of abnormalities in behaviour in the domains of "social development, communication, and repetitive behaviour/obsessional interests" (Baron-Cohen, Wheelwright et al. 2002). In this paper, the concepts of the empathizing and systemizing, together with the attributions 'male' and 'female', were presented. The empathising-systemizing theory is explained in terms of agency and intention: that people with empathising skills recognise mental states in others and produce "appropriate emotional response" and people with systemizing skills "understand and predict the behaviour of non-agentive events" (*ibid.* p495). And it is more than mere capacity – there is a recognition that capacity is fuelled by the subjects' own agency, their interests, desires and drive. The link to gender comes later in the paper and is based on Kimura's work (Kimura 1999) and work from his own colleagues, together with Asperger's previously mentioned conjecture.

Thus someone with autism is 'untuned' to the social world and Baron-Cohen and colleagues hypothesise that such social-tuning deficiency is a result of impaired empathising faculties. In a positive turn, Baron-Cohen and colleagues have considered the other aspect of the triad to do with repetition and obsession and constructed the concept 'systemizing' which for autistic individuals is "intact or even superior" (*ibid.* p495). Inasmuch as the systemizing concept comes from the criteria for diagnosing autism, one can see why Baron-Cohen would conceptualise systemizing as oppositional to empathizing. The work has a root in exploring Hans Asperger's 1944 notion that "autistic personality is an extreme variant of male intelligence" (quoted in translation from original German (Baron-Cohen 2003). Baron-Cohen's theory extends Asperger's notion of 'the male brain' as 'systemizing' (one that understands and builds systems) and he also develops a female counterpart by positioning 'the female brain' as 'empathizing'.

A BRIEF CRITIQUE

(i) The concept of empathizing as used in *The Essential Difference* is defined by means of the 'Empathy Quotient' questionnaire. In this questionnaire, empathy is construed sometimes to be social skills ("I can sense I am intruding, even if the other person doesn't tell me", "I find it hard to know what to do in a social situation") but also to relate to a personal emotional state ("it upsets me to see an animal in pain", "seeing people cry doesn't really upset me" …). This begs the question about what empathy is.

(ii) The notion that the concepts of systemizing and empathizing are negatively correlated is suggested but not stated; the nuance cannot be interrogated as the bivarate data are not presented. The book does not claim that systemizers are not empathisers; it is the discourse, rhetoric and style that presents this juxtaposition.

(iii) The book's essentialising of males as systemizers and females as empathizers based on statistics of the extremes, 'queers' the outliers and obscures the fact that the distributions of scores for males and females overlap considerably. Indeed, in another paper (Baron-Cohen, Wheelwright et al. 2001) Baron-Cohen's own data shows that for one characteristic, 'having attention to detail', that is associated with systemizing, the means of scores of 'control females' were actually higher than those of 'control males'.

(iv) Early on in the book, Baron-Cohen defines male and female in 5 different ways: (1) genetic, (2) gonadal, (3) genital, (4) 'brain type', and (5) 'sex-typical behaviour'. Definitions (4) and (5) are his own: your brain is 'female' if your empathizing is stronger than your systemizing, and vice versa, and sex-typical behaviour "follows from (4) brain type" (*op. cit.* 2003 p98). His definitions serve to reify his 'the essential difference' slogan. But tagging on to the list (1-3) of far more physically defined concepts of male and female, definitions that are based on a questionnaire, is misleading: it suggests that these concepts are more real than they actually are. The first three are based on bipolar physical states, not on overlapping normal distributions of attributes.

Issues related to mathematics

Mathematics involves systemizing: sorting out structural features of problems; understanding and working with the logic of events, machines or rules; representing ideas symbolically and attributing meaning to these symbols and operating with them creatively and independently. These are mathematical attributes that involve 'systemizing'. Other questionnaire based work from Baron-Cohen and colleagues, (Baron-Cohen, Wheelwright et al. 1998), certainly indicate that autism is more common in mathematically-orientated families. Highly talented Mathematics Olympiad winners surveyed by Baron-Cohen and colleagues had similar scores to an Asperger syndrome group on the 'attention to detail' sub-scale. On most other sub-scales they had scores about mid-way between the control and the Asperger group but were similar to the controls' scores on 'communication'. Empathy is easier with one with whom you co-systematise. And this suggests that communities of systemisers do develop inter-personal bonds that can be interpreted as empathy. It is worth noting that the notion of an 'appropriate emotional response' is very much culturally relevant. And so in the mathematics communities, if there is a lack of ostensive personal chat that should not be read as 'inappropriate emotional response'; quite contrary-wise, silent support of the practice may well be deeply felt, empathetic. Our maths students speak of the importance of their study mates who share their mathematics undergraduate practice.

DO WE SEE THIS SYSTEMISER/EMPATHISER DICHOTOMY IN OUR STUDENTS?

Our three-year longitudinal study of undergraduates' experiences (Rodd and Brown 2005; Rodd and Bartholomew 2006) included many extensive student interviews (n=93). Our interpretations of these interviews gave us insight into students' personalities, drives and conceptions of mathematics. In this section, we present extracts from interviews with three successful students . These particular students, who we have called Tessa, Janusz and Lindsey, were chosen as they illustrate variety in the way undergraduates expressed views related to their relationship with mathematics and to their social priorities. Thus they exposed something of their systemizing and empathising orientations as they are experienced in mathematics learning at university. Furthermore, each of these students achieved a top-rank (first class classification) after three years and thus could be said to exhibit good systemising skills.

Tessa

> I just worked all the time, I mean, I had a busy social life and everything but every spare minute I was like handing in sheets. "I live with artists and people who really don't do anything" who don't seem to be able to relate "to me revising for twelve hours a day ..because [otherwise] I'll fail".

> When we did a group project and we really, it was me and all my friends who all got 90% and we all worked really hard and did this project about sequences I think it was. …And we worked really hard and made this lovely presentation and I remember feeling really good when we presented it and we did well. And that felt quite good to be able to understand what I was saying and to speak to a class of maths students and know what I was talking about … [the lecturer] seemed really interested.. Oh, and we'd found something he didn't know. Nothing extraordinary but something he'd never noticed. So that was quite nice.

Lindsey

> Because I'm good at maths I like other people to be good at it as well. …I am very confident in explaining things … and I always seem to find a way to make it easier to understand, and that is what I like to be able to do for other people, to help them understand it as well.

> I want to be a maths teacher …it feels good, not just to learn new things for myself, but it feels good to be able to tell other people …to help [people] to understand those simple things, that is the same feeling as when you understand something more complicated itself. …If I become a maths teacher I'm going to have that feeling all the time.

Januscz

> I feel more in control of maths in a way, I know what I have to learn, I know when I've got it right. [I like] the structure of it [maths], the way you get a theorem and prove it systematically…my friends say 'oh maths! can you do this long sum for me? And it's not really about that for me, it's proofs and the ways you get around proving things

> when there is something I just don't know how you're supposed to get the answer to, it will trouble me a bit and then I can't sleep or I'll go to sleep and then wake up thinking about it

> I'd advocate a different method of assessment…I tend to slip back into trying to understand as much as I can every time it comes round to revise and that's where I lose out because I don't have that stringent discipline in just doing the [past] papers finding out exactly what I need to know and no more

> …maths isn't, for me it's not the be all and end all, I've always needed to have a social life…you're not a mathematician unless you dedicate your life to it almost

These three brief extracts indicate modes of empathising and systematizing maths undergraduates experience. They illustrate how social and emotional issues relate in different ways to the student's systemizing need. Thus the type of empathizing/systemizing dichotomy that Baron-Cohen writes about does not show up in our data (what is presented is representative in this regard). Women have always been systemisers, but traditionally in different domains to men. Baron-Cohen's questionnaire to gauge a person's systemizing quotient (209-16) is not written to pick up female systemizing. Culturally male activities like doing electrical wiring (SQ qn 7) are given as prompts to assess systemizing skills and in the item on cooking, arguably a gender-balancer, the statement refers to 'a final product' (of cooking) rather than to a 'meal' or 'food' thus presenting the potentially female-friendly item in a female-alien discourse which is missing from the prompts in Baron-Cohen's questionnaire. So it comes as no surprise that males score higher on his test; Furthermore, females' mathematical achievements, at least in England, are increasing (QCA 2004); there is also indication that girls don't necessarily see maths as a boys' subject (Francis 2000) though this has not been a universal change. Females do successfully participate in mathematics though this maybe relatively 'invisibly'.

This critique of *The Essential Difference* has been presented in order to draw teachers of undergraduate mathematics attention to the work that positions males and females as essentially different and that this difference hinges on a personality trait related to systemizing – a trait that is intrinsically mathematical. Teachers of undergraduate mathematics students may want to question this positioning, the assumptions that underpin the work and interrogate the notions from their own experience and identity. What could be an alternative? Personalities can be more or less geared to precision, systemizing, logical reasoning, single-mindedness, etc. They can be more or less geared to other attributes too: compassion, social awareness, fashion sense, family bonding, wide attention span. Specifically, considering the notion of 'systemizing' that is used in the book, we can ask whether this notion of a systemizing mind is helpful in selection? for guidance of students? or for pedagogy?

CONCLUSION

The notion that there are empathising and systemizing skills that psychologists can measure through testing could be a useful contribution to our understanding of peoples' skills and propensities. Yet Sheila Greene (Greene 2004) behoves us to be wary "naïve biological thinking": she positions this work of Baron-Cohen as part of a new version of biological determinism where gender-linked dispositions are presented as “differences not deficiencies”. Her view is consonant with ours that the ploy that affixes the epithet of male to one attribute, here systemizing, (even while saying ‘it’s needn’t be you’) is to stain the discourse of mathematical participation and achievement. Inasmuch as these ideas become culturally familiar through media coverage they self-reify. Inevitably, consciously or subconsciously the message filters in ‘maths is not female’. This message has historical resonance, after all there were very, very few professional female mathematicians until 1960s feminism opened some doors and there are still not many. But women have always been systematizers: walk into kitchens, nurseries, Women’s Institutes and in suitable examples of these female-privileged domains you’ll be able to see as much systemizing as in car mechanics’ workshops.

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A CULTURAL-HISTORICAL APPROACH TO TEACHING GEOMETRY PART 2: THE RESULTS OF A PILOT STUDY

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In the last proceedings (Rowlands and Carson, 2006) we discussed a curriculum initiative that aims to 'bring to life' the major primary events in the history of Greek geometry. In particular, the combination of the intellectual act of abstraction and the possibility of formalised, logical proof were discussed. In Part 2 the results of a pilot study to see whether year 9 'mixed ability' and year 10 'gifted and talented' students can be meaningfully engaged with these two primary events are discussed.

INTRODUCTION

In the last proceedings we outlined a curriculum initiative that aims to guide students through the processes of abstraction and proof as two 'primary events' ('key ideas') in geometry's developmental history. With the first primary event, students would be guided through 6 levels of abstraction and for each level the class would discuss and explain what has been carried forward and what has been left behind, culminating in discussing the Forms and the possibility of proving opposite angles are equal.

This article presents a summary of a pilot study to evaluate the likelihood of successfully implementing the two primary events with secondary school students. The extent to which the students engaged with the activities would be a measure of success. Because the author was the teacher the account given below is part subjective, but the extent of the engagement was verified by observers.

The pilot study consisted of one R. I. Maths Masterclass session comprising of 36 year 10 students and two Widening Participation sessions comprising of 'mixed ability' year 9 students. The next section explains the rationale behind the pilot study, followed by accounts of the Masterclass and Widening Participation sessions.

THE PILOT STUDY

This initiative was designed for 12 or 13 year old learners of 'average' development. An evaluation of such an initiative would need to take the form of an action research project involving the implementation of the initiative as a course taught within the classroom setting. The purpose of the pilot study was to see whether learners could be engaged in such an initiative and to what extent. The Masterclass and Widening Participation sessions were two opportunities to pilot the implementation of the two primary events but there were three major problems to consider:

The context, whether it is widening participation or masterclass, is artificial compared to a course in geometry within the normal classroom setting. However, the artificiality of the context makes it *more* difficult to introduce the two primary events. Within a normal classroom setting the introduction of the two primary events would be perceived by the students as part of the course, whereas in Widening Participation the expectation by the participants may be higher and, given this is not a course in geometry, the whole point of the exercise may be more difficult to grasp.

The learner's previous experience of geometry lessons may influence their engagement with the topic under evaluation. 'Gifted and talented' learners may have already been exposed to deductive proof, so introducing it might go against their expectations, with the whole exercise appearing superfluous.

The content and 'delivery' may not be appropriate to the learner's development (or 'ability'). 'Gifted and talented' 15 year olds may be cynical about going through the 6 levels of abstraction, since they are already very capable of thinking in the abstract.

To overcome all three problems it was decided to engage the class with the nature of proof before taking them back to the historical moment when the act of creating ideal objects made proof possible. This was achieved by beginning each session with the question 'the three angles of a triangle add up to what?' followed by 'how do you know?' In each session there were facial expressions of what may be described as 'cognitive conflict' with the second question. This was followed by discussion and an active engagement in proving formally the angle property of the triangle. This would set the context for what was to follow: an introduction to Thales and the importance of proof. By starting with proof it was hoped that the context and relevance of the two primary events would be provided, hence overcoming the first problem. As for the second problem, if students were already familiar with geometric proof then this would be revealed in the discourse, but hopefully in such a way that its importance could be discerned and thus not appear superfluous – thereby reintroducing proof rather than 'mistakenly' introducing it. If proof is reintroduced then the significance of the 6 levels of abstraction could be placed in the context of how proof was first made possible, hence relinquishing the third problem. As it turned out the learners were not familiar with proof, including the Masterclass.

THE MATHS MASTERCLASS SESSION

The duration of the session was one and a half hours. I began by stating the format of the session and the ground-rules for discussion, such as listening to each contribution, not to ridicule the answers given by others, etc. I then asked the question 'the 3 angles of a triangle add up to what?' and received the reply 180. I followed with 'how do you know?' For a few moments there were signs of cognitive conflict. Someone mentioned the tearing of the angles demonstration but a student responded that the demonstration only applies to the triangle torn, to which I added 'what of all triangles?' Another student suggested measuring the angles, but two students responded that the measurement would only apply to the triangle measured. I also added that I spent last Saturday night measuring the angles of a 100 triangles and only twice did I measure 180.0, with the mean average 180.7.

I asked how we can show the angle property for all triangles and someone suggested constructing a line parallel to the base of the triangle drawn on the board so that the angles of the triangle can be related to the angles that make up a straight line. I demonstrated the proof as he said it, adding that we can assume that alternate angles are equal when a line cuts two parallel lines. I asked whether he had seen the proof before but he said he could not remember. We then discussed the nature of this proof with respect to all triangles and I mentioned and explained the term ‘logical necessity’. I also explained that proof forms the heart of mathematics.

This was 20 minutes into the session. At this point I formally introduced the session and outlined what we were going to do. I then introduced the class to the Egyptians, their practical geometry and Thales’ contemplation in the desert of ideas existing above and beyond the observed configurations that once served to represent those ideas. I then took them outside to sketch the four stakes and two intersecting ropes and to take them through the 6 levels of abstraction.

One has to bear in mind that this is not in school time and hence compulsory. Nevertheless, out of the 36 students, 35 sheets were handed in. 11 were highly detailed, with notes and comments written in earnest next to the drawings. You can sense pride in these notes, with 6 stating that they want the notes/drawings back. Although it was made clear that these sheets will be collected (and returned if desired), the notes are in a sense personal and not constructed for evaluation by the teacher. 15 of the remaining sheets contained the necessary drawings accompanied with relevant notes but the commentaries concerning level 6 were scant. The remaining 9 contained diagrams and notes but were in some ways incomplete. Interestingly, one sheet from this sample was unique in constructing a parallel line to prove that opposite angles are equal using alternate and corresponding angles.

The following is a sample taken from the sheets. This is a small sample because much of what has been stated has been expressed (albeit in different ways) by others. Of course, much of what was written was prompted by discussion.

Level 1: *Literal representation*

‘Leaving behind angles, measurement.’

‘Shadows, grass, tone, daises, flies, detail, colour and 3 dimensionality have been left behind while the ‘shape’, the outline, the essence of the image remain.’

‘Left behind: 3D, colour, reality, texture, actuality, physicality, daises. Taken: idea, basis, images, impression, representation.’

‘Drawn forth: rope, tree, stakes, mud, grass, daises [the detail in the accompanied drawing shows each, including the texture of the rope]. Left out: building, traffic, road, people.’

‘Taking a basic idea of the image and abstraction, but the actual configuration is easier to see and you are leaving behind the 3 D image.’

‘Can only be seen from your point of view. Can’t see it from a different angle.’

‘We are taking an image representing how our eyes perceive what is before us’.

Level 2: ***More abstract representation***

'Left behind the physical nature of the object. Taking the shape with you.'

'Now the image is left behind and only the position and presence of the stakes and rope remain.'

'Taken: shape, idea. Left: reality, size, perspective and above [referring to level 2].'

'Drawn: the distance between the points. The points of attachment to the ground. Left: everything else'

'This is a more basic symbolic view and the actual thing is more graphic.'

Level 3: ***Model***

'Allows freedom of sight.'

'Now the size is left behind and some of the appearance, but scale and everything else of the original stakes remain. It can be seen from different perspectives, so it is superior to the 1st level of abstraction.'

'Able to see anyway we like.'

'Drawn forth: the straightness of ropes and stakes.'

'More perspective. Less actual representation. More accuracy.'

'Rotation can be done.'

'We take something similar to reality, We can perfect reality, proportion'.

Level 4: ***Private Concept***

'No sticks, no dots. General idea of 2 intersecting lines – no specifications, no measurements.'

'All you take is the general idea of two intersecting lines. It is general because there is no specificity about the length of lines, where they cross or the angles at which the lines cross. All we have is that they do, at some point, cross.'

'Imagination – 2 lines intersecting – no sticks – general idea - no specifics (where lines cross, length, angles) – no scale – just 2 lines.'

'The intersecting lines remain; everything else is lost. Angles in mind are immeasurable, as are lengths – they can be changed at will.'

'Different to everyone elses. Cannot be measured. Can be changed.'

'No one can see it. It's in my mind'.

Level 5: ***Authorised concept***

'The concept that has been established by a community, in this case mathematicians.'

'Not necessarily true (e.g. world being flat).'

'Agreed concept found in textbook'.

‘- agreed upon by mathematicians – in text books etc. – established by a community. (Is something true because its in a textbook? Is something in a textbook because it is true?) Just because people think something does that make it true? Belief does not make something true. For something to be really true we need proof. What is the nature of truth? Thales (first person to prove anything) asked how do we prove opposite angles are equal? Opposite angles are equal because of logical necessity, it cannot be any other way. It’s not possible for them not to be true.’

Level 6: ***Platonic Form***

‘The idea is always true even though it may not ever of happened. The idea which is totally independent of human minds.’

‘What is truth? Truth is something abstract that has no place; it is everywhere at all time. Something does not need to be proven to be true – it is true already, but we have to prove it to know as humans that it is true. Because potentially the triangle could have existed, it is true, although it has not necessarily happened.’

‘These forms are more real than physical reality! Did they exist before humans. Humans <u>discovered</u> it they did not <u>invent</u> it, so it must have existed before – Plato’s argument. The concept can exist without anything else.’

‘Truth doesn’t depend on human agency, the truth is what is real. . . . Whether people are still around, things will still be true even if there is nobody to say it is true’.

‘Not opinion, not consensus’.

‘Real truth lies beyond what we can perceive as truth.’

‘Where does this immortal truth reside? Its just there. Everytime 2 lines intersect. It’s independent of whether there are 2 lines that intersect. (‘Platonic Form’ – named after Plato circa BC 450). Potentially true. There was the potential for triangles before humans, but no way of knowing if there were. Concept of triangle independent of humans. Humans make things an actuality – potential for anything. Thales changed geometry from practical to concept – objects in the mind. Only then can properties be explored. Ideas that are abstract and concept, actually more real than physical objects.’

After level 5 and for the remaining 30 minutes the class went back into the classroom for a discussion on the Forms leading to proof. At this stage there was very little note taking as all the students seemed fully engaged with the discussion. The discussion developed into whether mathematics is invented or discovered. At some stage I led the class through a formal proof concerning opposite angles by asking questions. I then asked them who was it that actually performed the proof, to which a student stated ‘we did’, followed by smiling faces. This was a cue to introduce the slave-boy in Plato’s Meno, who under Socrates’ questioning was able to prove Pythagoras’ theorem without having any previous mathematical experience, raising the question as to how the slave-boy knew the answers to the questions. I concluded the session by placing the Forms in their cultural-historical context.

YEAR 9 WIDENING PARTICIPATION SESSIONS

These consisted of two groups of about 30 students each. The session for each group was half an hour less that the Masterclass session. The session format would essentially be the same, but the 'mixed ability' of the class and the shorter session meant a focus on the 6 levels of abstraction and the opposite angle proof.

After preparing the groundwork the sessions began with the two questions. On both occasions there were suggestions of measurement and the tearing demonstration. Both groups followed the discussion on the limitation of these suggestions and were able to respond to questions leading to the formal proof after I suggested the construction of a line parallel to the base. I praised both groups for their ability to prove, told them the importance of proof and briefly introduced them to Thales.

Both groups went outside to undergo the 6 levels of abstraction and all took the exercise seriously. When everyone closed their eyes there was no silly behaviour.

There was, however, very little or no commentaries on the sheets. This is perhaps to be expected as this was a younger group of 'mixed ability'. Nevertheless, many of the drawings were of considerable detail. The students were able to give examples of what was carried forward and left behind for levels 1 to 4.

The discussion of level 6 was restricted to the existence of geometrical objects, leading to the opposite angle proof. By their engagement with level 6, the majority of students appeared to follow the idea that a geometrical object is not something we can draw and were able to answer questions concerning the dimensions of geometrical objects, whether or not we can see them and whether they existed prior to humans. They were able to answer questions leading to the opposite angle proof.

CONCLUSION

The aim of the pilot study was to see the viability of engaging secondary school students with the two primary events leading to the nature of proof. Was the study successful? A 'yes' to the question can be asserted on the grounds that the level of engagement by each group was consistently high, verified by observers. The sessions could have been disasters, with students regarding the exercise pointless with low level disruptive behaviour as a result. The active participation of all the students was visible and was not a result of politeness to the occasion. This prompts the next stage in the research, comprising a more objective evaluation of the initiative. This would involve the observation of a series of lessons within the classroom setting coupled with an evaluation of learning outcomes.

Of course, any evaluation of the project will involve the extent to which the teacher can engage the class as it would the implementation of the two primary events. The ability to pose the appropriate question at the right time so as to lead the class to the desired outcome, the ability to handle diverse responses and the ability to captivate using historical narrative are examples of skills that can be developed and formalised as an outcome of evaluating the initiative.

MECHANICS SHOULD BE INTEGRAL TO SECONDARY SCHOOL MATHEMATICS

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Mechanics has never been the most popular subject in A-level mathematics, either with the students, the teachers or educators. The 'innovative' attempts to popularise mechanics appear to have failed and it is conceivable that the subject will be dropped from the A-level syllabus within the next two decades. This article argues the importance of mechanics and why it should be integral to secondary school mathematics. Mechanics is the exemplar of mathematical modelling, is the logical point of entry for the enculturation into scientific thinking and provides the means to develop an understanding of the relationship between mathematics, the theoretical objects of science and the way science and mathematics speak of the world.

INTRODUCTION

Once considered boring, highly theoretical and hence irrelevant (Berry, 1990), mechanics still isn't a popular subject despite the attempts to make it relevant. Since the radical change in the A-level syllabi (now syllabus) during the late 80's and early 90's, when modelling was introduced in mechanics to make it relevant, there has been a decline in the proportion of A-level students doing mechanics (Kitchen et al. 1997). Although A-levels were changed to improve student motivation and success rate (Kitchen and Williams, 1993), with mechanics this has not been successful. There are many complex reasons for this, such as having to teach the way mechanics has been structured by the examination boards and their associated textbooks for assessment purposes. Yet despite its treatment and the straightjacket of having to teach it that way, mechanics as a subject in applied mathematics is so important that consideration ought to be given in treating it as a central topic in mathematics prior to the Sixth form. This article attempts to show the importance of mechanics both in terms of its logical character and the corresponding educational benefits.

It is perhaps safe to say that the unpopularity of mechanics extends to a large proportion of mathematics educators. Ironically, the same mathematics educators might agree on the virtue of modelling. The irony is that mechanics is the exemplar of mathematics modelling. In terms of its logical character this paper attempts to explain the importance of mechanics in modelling and how this offers the opportunity for even the most concrete thinkers to think not only in the abstract but to think how the abstract can model the world. There is a dual but related sense of the abstract here: the contemplation of (im)possible worlds (thought experiments) essential in explaining and modelling the physical world and how the theoretical objects of science and applied mathematics speak of that world. Such contemplation requires imagination (examples may include a world of no gravity, a frictionless surface, etc.) that is rule-governed by the logical character of mechanics. This paper offers a brief description of that logical character and attempts to explain how this character not only makes mechanics the central subject in science but also applied mathematics. What may initially seem contrary to the title, this article begins by situating mechanics in science so as to reveal its logical character, leading to the reasons why mechanics should be integral to the secondary school mathematics curriculum. Further reasons are then discussed, including mechanics as the exemplar of how research can inform practice.

THE LOGICAL CHARACTER OF MECHANICS AND ITS IMPORTANCE IN SCIENCE EDUCATION

There are many reasons for teaching mechanics as a central topic in science. For example:

- The history of the scientific revolution of the Seventeenth Century *essentially* began with mechanics. Essentially because the emergence of Seventeenth Century mechanics was paradigm changing: from the Aristotelian (cosmological and implied mechanical) world view to the modern one.
- Children can be introduced to mechanics without having prior knowledge of science. To contemplate the forces acting on a thrown ball, for example, requires no prior instruction.
- Mechanics is the logical point of entry for the enculturation into scientific thinking and is not merely a domain of physics that shares its place amongst many (Carson and Rowlands, 2005). This leads onto the next point:
- Mechanics is concept forming.

The last two points are the most important. Mechanics is not merely a topic amongst many but structures the very edifice of science. For example, without mechanics there would be no electromagnetism or kinetic theory of gases:

> Mechanic determines one form of description of the world by saying that all propositions used in the description of the world must be obtained in a way from a given set of propositions – the axioms of mechanics. It thus supplies the bricks for building the edifice of science (Wittgenstein, 1974, Proposition 6.341).

> Mechanics is an attempt to construct to a single plan all the *true* propositions that we need for the description of the world (Wittgenstein, 1974, proposition 6.343).

The basic interaction variable in mechanics is force (Hestenes, 1987) and so it follows that, as stated by Kitchen et al. (1997), Newton's laws should be a central component of student understanding of science. According to Galili (1995):

> A huge edifice, which today we call physics, consists of various domains. *The importance of mechanics is more than just being one of these domains. It determines the 'rules of the game', defines the main tools in physics, presents the most universal laws of nature. It actually describes the method of the discipline of physics which is then applied in all other domains in this discipline. This is why mechanics always opens any physics curriculum.* (Galili 1995, p.371, emphasis added)

Mechanics is concept forming because the axioms define force from which work can be defined and in turn the concepts of energy can be derived. The conservation laws rest upon the laws of motion. This is how the axioms supply the bricks for building the edifice of science (see Rowlands, 2003; Carson and Rowlands, 2005).

REASONS FOR TEACHING MECHANICS IN MATHEMATICS

Mechanics is an exemplar of mathematical modelling because of its concept forming character. A field may be modelled as a rectangle, the properties of the rectangle may be applied and a desired result calculated, but there is no scientific component that explains anything in the process. This is not to undermine mathematical modelling in general, but mechanics is a form of mathematical modelling that is able to explain phenomena as well as yielding the relevant data. That ability to explain is due to its logical character, giving rise to the relevant concepts. The following are reasons for why mechanics should be integral to the mathematics curriculum:

- Modelling in mechanics has primacy over other areas of applied mathematics (MA, 1965). It is the essential ingredient in understanding modern domains and is the only satisfactory vehicle for demonstrating applied mathematics as a discipline in its own right (Crighton, 1985). This may explain why "attempts to teach modelling in areas other than mechanics in A-level have, so far, proved inadequate" (Kitchen et al. 1997, p. 166).
- A mechanics course could (and should) provide the 'linking framework' to other areas of applied mathematics, such as differential equations which expresses the principle of many sciences and can begin with dynamics (MA, 1965).
- Unlike other forms of mathematics such as statistics and decision mathematics, mechanics provides the perfect opportunity to become fluent in algebra, trigonometry and calculus (Kitchen et al. 1997). Conversely, however, one of the few satisfactory ways in presenting the triangle law of vector addition in pure mathematics is by way of the parallelogram law of the addition of forces. This can be done using the Leeds mechanics kit. This is not to undermine the importance of other forms of mathematics, but mechanics provides the perfect opportunity to comprehend as well as to utilise pure mathematics.
- Modelling in mechanics has a 'scientific component' whereby the quantities modelled are well-defined, unlike other domains, such as economic theory (Crighton, 1985). Mechanics contrasts with the *pragmatic* models of finance management and population models which are "characterised by a purposeful avoidance of those fundamental concepts that the true mathematical model seek to describe. . . They seek to solve a problem and invariably omit any conceptualization procedure" (Hickman, 1986, p. 734).

This contrasts with what Clement (1982) described as the assumed modelling methodology, which is apparently structured by the 'scientific method' that begins with observation and is followed by hypothesis and validation. But modelling in mechanics is a process whereby an explanation is sought by formulating an appropriate model (Hestenes, 1992). As such mechanics not only provides an opportunity to understand the nature of science (NOS) but also theoretical modelling as compared with empirical or pragmatic models. NOS is not a clip board of recorded data from which relations are established, nor is theoretical modelling.

Compared with data collection, theoretical modelling engages different cognitive processes and research can inform us as to those cognitive processes and the teacher as to the likely cognitive state of her students. This is discussed next.

MECHANICS AS THE EXEMPLAR OF HOW RESEARCH CAN INFORM PRACTICE

Conceptual change is the largest area of research in science education and the biggest field of research in conceptual change is mechanics, particularly force and motion. This isn't surprising if we regard the importance of mechanics in structuring other domains in physics. Force and motion is not only fundamental with respect to the edifice of science, but also with regard to understanding learning processes, whether we frame those processes in terms of cognition or socially negotiated meaning. Mechanics can provide insights into the ways we think and learn. For example, from abstract considerations, such as thought experiments, to the way the abstract can enable learners to model real situations. Experience has shown that mechanics can also provide a means by which the concrete thinker can think in the abstract – in imaginative but rule-governed possible worlds. Research in these domains can inform the teacher the kind of intuitive responses she can expect from asking questions concerning force and motion and the strategies that can be employed in directing the mechanics class in understanding force and motion.

For example, in 'Widening Participation' sessions whereby year 9 'mixed ability' students are introduced to 'probably the most important question in science', students are asked to identify the forces acting on a vertically thrown ball going up. Research has shown that many students will identify a force pushing up along with gravity and air-resistance, and you can almost guarantee that many and if not most of the students in these sessions will do the same. The strategy adopted is to say that this answer is not the correct answer but it is a *good* answer because that was the answer given by Aristotle and accepted for two thousand years until challenged relatively recently by Galileo. The students are then engaged with the history of the subject which can put into perspective their own intuitive responses to force and motion. Mechanics provides the opportunity for research to inform practice.

DISCUSSION

In arguing for a broad mathematics curriculum at A-level and against the 'more specialist mathematical options at 16-19', Margaret Brown states

> The position of mechanics warrants special attention. Although it is a valid and important application of mathematics it should appear in the A-level mathematics course only as

> one of many, and should not have the favoured place that it currently occupies in England and Wales (and in no other country). Mechanics should properly be part of an award in engineering or physics. (Brown, 1999, p. 85).

Well before 1999 mechanics had lost its favoured place. Since the introduction of the modular scheme, many and if not most A-level mathematics students do not do M1 (the first module in mechanics) and very few do M2. Despite the recognition of the importance of mechanics in A-level mathematics, there seems to be the implicit desire to relegate mechanics to physics or engineering. Brown's argument is that 16 year old students should be given a wide choice of applicable mathematics at A-level because many of them are unaware of what career paths to take or what subject choice to make. Within the context of arguing against design options in accord with university course requirements, Brown states that 'university engineers may favour mechanics over statistics to ease their own teaching, but fail to recognise that statistics is as important a tool for engineers in employment as it is for other professionals in industry and commerce' (Brown, 1999, p. 85). Is the inclusion of mechanics really to do with 'ease of teaching'? Without undermining statistics, the converse could be argued but with a greater imperative: mechanics, because it is the exemplar of mathematical modelling, would benefit even those who go on to choose economics. Besides, to expose pre-16 year olds to mechanics as a subject in mathematics would go some way in influencing what subjects to study in mathematics at A-level. Such exposure would help develop the ability to model in mathematics as well as developing an understanding of the nature of science. Exposing pre-16 year olds to 'Data Handling' instead may well be superfluous as everything covered can quickly be taught in post-compulsory education.

That mechanics has a 'favoured place' in England and Wales is not a satisfactory argument in downplaying mechanics. The split between 'pure' and 'applied' is only just over a century old and many leading mathematicians of the past were also physicists and vice-versa. For example, Euler's mechanics for the rotation of a rigid body, Gauss's work on electro-magnetic forces, Galileo on probability. The point is, mechanics is very much a subject in mathematics as it is science and should be respected as such.

One module in mechanics is not enough to appreciate mechanics as a unified whole and there is an irony in all this: presumably M1 provides the basis for understanding mechanics (force and motion), yet after learning this 'difficult bit' students then take a module in statistics or decision mathematics. The irony is that once force and motion is understood then there is a basis for understanding all the various topics in mechanics, such as projectiles, circular motion, periodic motion etc. All these topics can be treated as the application of the laws of motion, but if M1 is the only mechanics module taught then the *significance* of M1 is lost. Mechanics needs to be treated with sufficient depth and that implies a linear course of half an A-level. This is unlikely given the status quo.

The whole argument presented above is not only a pedagogical issue but also a scholarly one. But as one commentator stated ‘deep philosophical arguments do not cut much ice with the increasingly pragmatic world of school mathematics’ (see Rowlands 2003). That world is dominated by the imposition of the state (QCA, formerly SCAA) and the examination boards ready to acquiesce in the pursuit of franchise. There has been scholarly criticism of the way the exam boards have tried to make mechanics practical, relevant and interesting, but unfortunately no response has been forthcoming. Arguments for a linear course in mechanics for all the reasons argued above may well fall on deaf ears, but change is possible. If there is a possibility for M1 to be redesigned by way of introducing thought-experiments, taking into account student misconceptions of force and motion, framing the topic historically and constructing a basis for understanding the subject as a whole – in short, a module that engages the learner with the logical structure of the subject and how that structure models the world - then there might be the demand for further modules in mechanics.

The attempts by the state and the exam boards to reform mechanics have failed and if the situation continues then there is every danger that mechanics will be dropped from A-level mathematics. If this does happen then arguably that would be a great loss to mathematics education. To prevent this from happening as *fait accompli* there must be scholarly discussion and debate and in a way that the issues cannot be ignored by those in positions of power regarding the curriculum. Practitioners, who are at the mercy of state diktat and exam board entrepreneurs, could well empower themselves by partaking in arguments concerning the curriculum and expressing the message: ‘contrary to the way we have been treated, we also have a say!’

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THE LONG-TERM EFFECTS FROM THE USE OF CAME (*COGNITIVE ACCELERATION IN MATHEMATICS EDUCATION*), SOME EFFECTS FROM THE USE OF THE SAME PRINCIPLES IN Y1&2, AND THE MATHS TEACHING OF THE FUTURE

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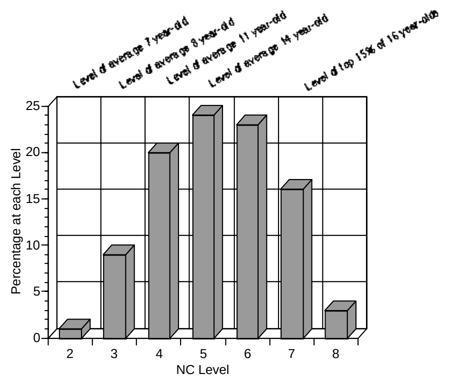
The CAME[1] *project was inaugurated in 1993 as an intervention delivered in the context of mathematics with the intention of accelerating the cognitive development of students in the first two years of secondary education. This paper reports substantial post-test and long-term National examination effects of the intervention. The RCPCM project*[2]*, an intervention for the first two years of Primary education, doubled the proportion of 7 year-olds at the mature concrete level to 40%, with a mean effect-size of 0.38 S.D. on Key Stage 1 Maths. Yet, instead of the intervention intention, it is now suggested that a better view is to regard CAME as a constructive criticism of normal instructional teaching, with implications for the role of mathematics teachers and university staff in future professional development.*

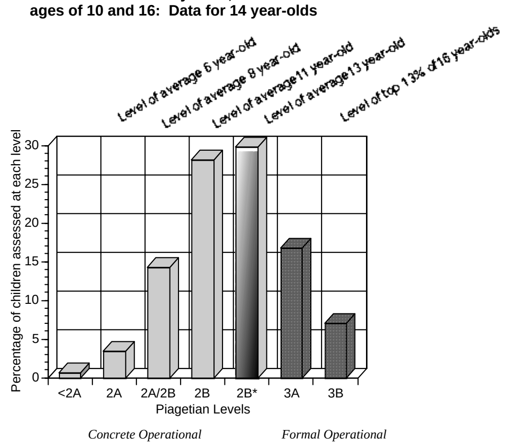
BACKGROUND TO CAME AND RCPCM

In the mid-70s CSMS[3] survey 14,000 children aged 10 to 16 were given three Piagetian tests to assess the range of thinking levels at each year. Figure 2 shows the findings.

Key Stage 3 National Statistics for Mathematics 2000: all 14 year-olds



1974/75 CSMS survey of 14,000 children between the ages of 10 and 16: Data for 14 year-olds



By 14 only 20% were showing formal operational thinking (3A&3B). This mattered at the time because current O-level science and maths courses, designed for grammar-school children in the top 20% of the ability range, required this level from the end of Y8. In the 80s the *Graded Assessment in Maths* scheme for the ILEA found that by the age of 12 the children's mathematics competence had a 12-year developmental gap between the above-average and those at what would later be National Curriculum Levels 1 and 2.

In 2002 the Government's own Key Stage 3 statistics for maths showed the same spread, as can be seen in Figure 1. When RCPCM conducted their first Pre-tests on 5 year-olds' classes they found comparable wide spreads in cognitive development.

The intention of both Projects was to increase the proportion of children able to benefit from good instructional teaching.

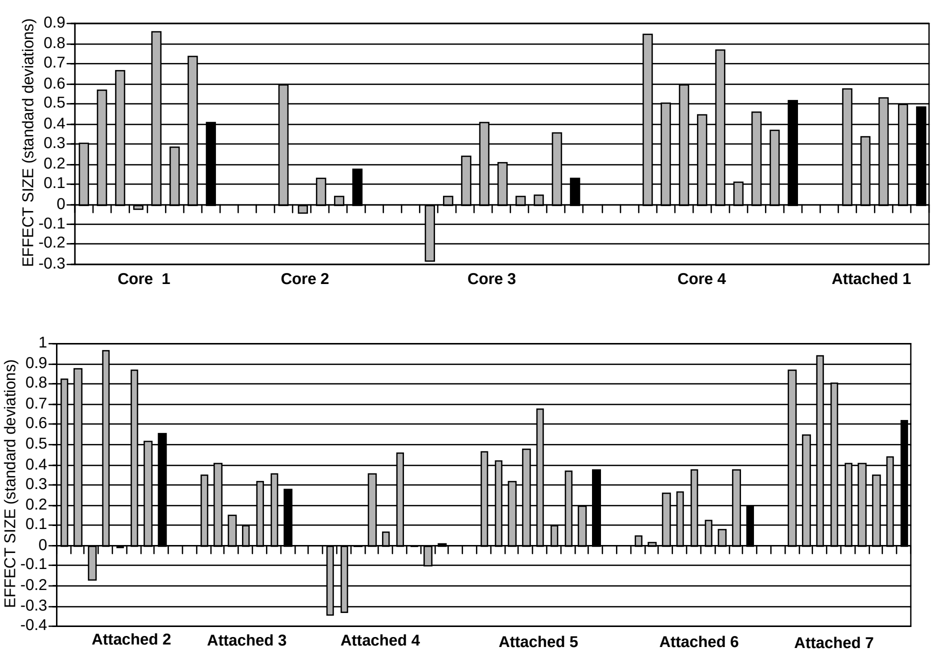
INTRODUCTION

The theory and practise of the CAME project is discussed in detail in Shayer & Adhami (2006). Essentially it takes a cognitive approach to the learning of mathematics, but integrates this with collaborative learning challenging the students initially in the Thinking Maths lessons (Adhami, Johnson & Shayer, 1998), with the teachers then using some of the same teaching skills in their other maths lessons. Space allows only for the results to be reported here.

EFFECTS OF CAME

The CAME project used the Thessaloniki Maths test (Demetriou, et al.1991) as a Pre-test in Y7, and as a Post-test at the end of Y8. Figure 3 presents the effect-sizes of each class, in relation to the test norms as controls, shown by the test. The black bars are the schools' mean effects. The Core schools were those which Shayer and Adhami were able to visit frequently: the Attached schools received only PD at King's.

Figure 3: CLASS GAINS ABOVE EXPECTED GAIN OVER TWO YEARS



No-one used to working with whole school departments will be surprised by the individual variations in classes within each school, or indeed, between school and school! It is however noteworthy that Attached school 2, having the largest mean effect-size, had one class with an effect-size near 1 S.D, and another with a zero effect. They were both taken by the same teacher: the one with the zero effect was a remedial class. The interpretation suggested is that for collaborative learning to work in a Vygotskian way, the remedial class lacked any pupils of intermediate ability that might provide for them the higher level insights enabling them to complete their individual zones of proximal development (ZPD).

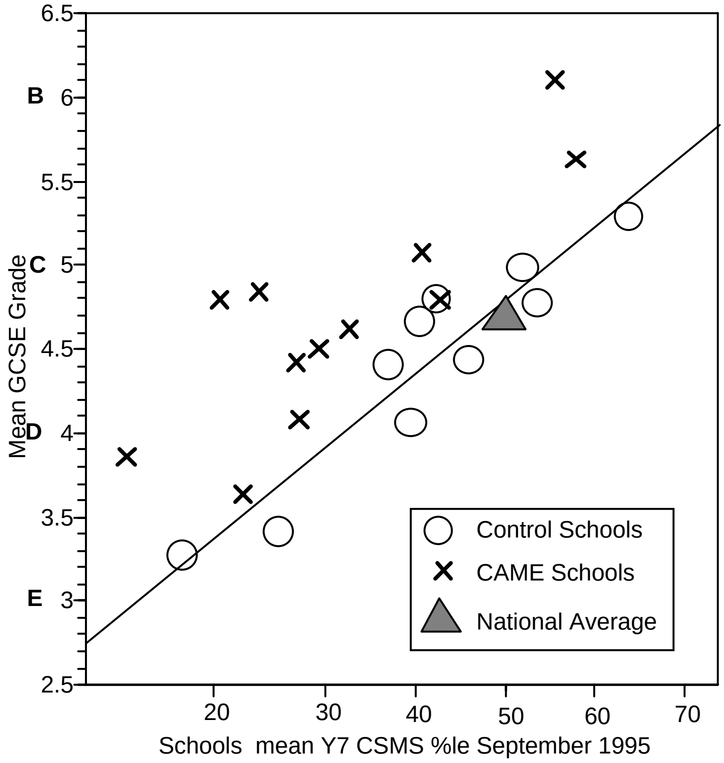


Figure 4 shows the long-term effects on GCSE 2000. The Control schools had either the Thessaloniki Maths test, standardised to the CSMS norms, or another CSMS test, as Pre-tests in Y7. It can be seen that the regression line for the Controls passes very close to the National average.

Comparable effects were found in both GCSE Science and English, as shown in Table 1. This provides the evidence that the general learning ability of the pupils was affected by the CAME intervention, as well as their maths achievement.

Table 1: Effects of the CAME Project.

	Mean Effect-sizes
Thessaloniki Maths Post-test	0.34 S.D.
GCSE Maths (2000)	0.44 S.D
GCSE Science	0.30 S.D.
GCSE English	0.32 S.D.

Finally, Figure 5 shows the correlation between the 1997 Post-test effects, and the GCSE 2000 effects. This is the evidence that the long-term GCSE achievement of the students is related directly to the intervention in Y7/8.

EFFECTS OF THE RCPCM PROJECT

The theory and practice of RCPCM is fully presented in Shayer & Adhami (2003).

The RCPCM project was designed to build on the expertise already developed in the research project CASE@KS1.H&F[4] (1997-2000). Children in Y1, in groups of 6 were given, for about half an hour every week, interactive and collaborative learning focused on the major concrete operational schemata described by Piaget.

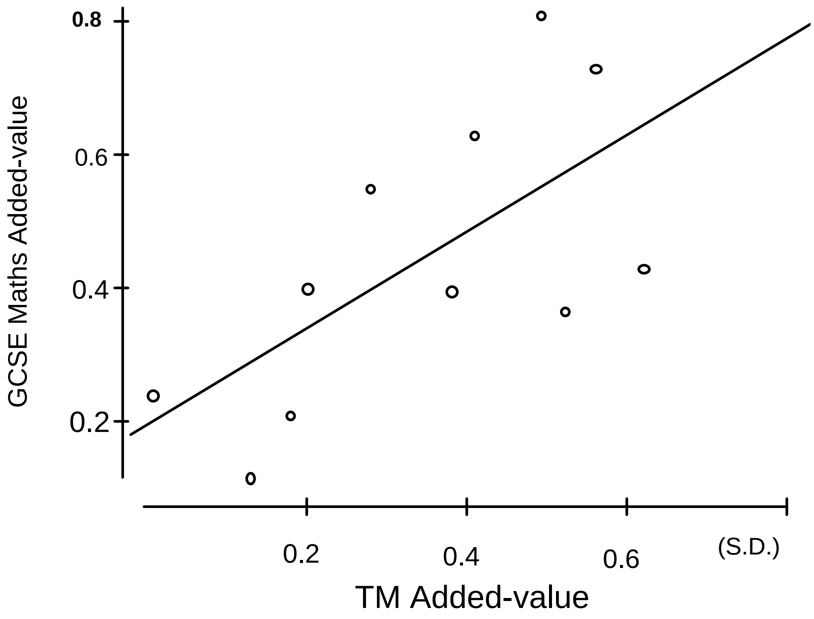
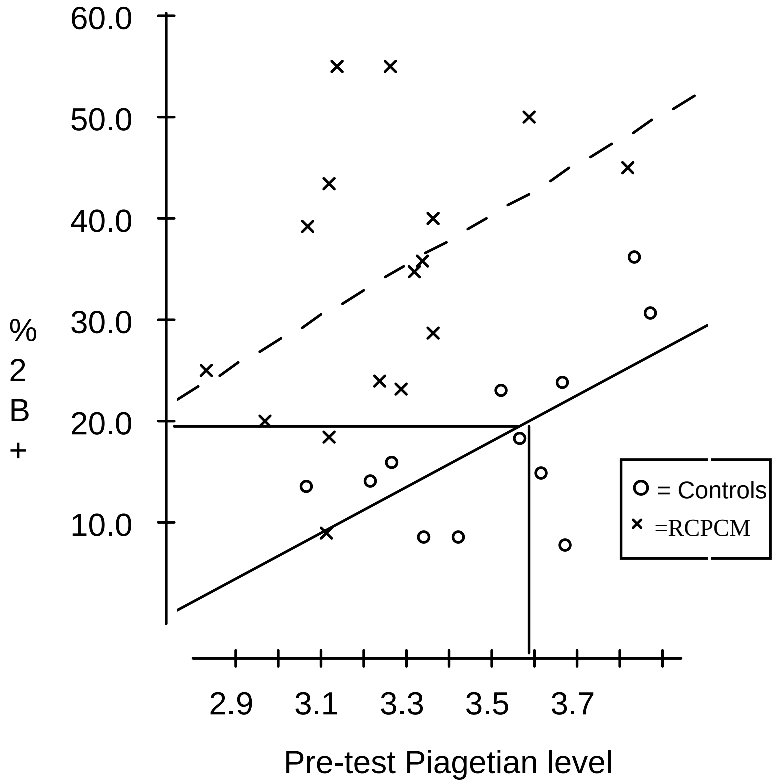


Figure 5: Correlation between gain scores of GCSE 2000 Maths and Thessaloniki -Post test 1997 both in relation to Thessaloniki Pre-test 1995

By Friday each week all the children in the class would have participated in the week's task. Unlike the original CASE project for Y7/8 the activities were not placed within the context of any particular school subjects. The effects of this one year intervention are reported in Adey (2001). Using Piagetian Pre- and Post-tests, effect-sizes, in relation to Control schools, of 0.47 and 0.43 standard deviations were obtained. The lessons and materials used have now been published as *Let's Think* (Adey et al. 2001).

The method proposed in the RCPCM project was twofold. In Y1 'Thinking Maths' lessons (*TM*) would be designed in such a way that the teachers could be led to manage the children's collaborative learning in the context of maths using similar skills they were also using in their *Let's Think* activities. In Y2 the *Let's Think* work would have ceased but the children, already used to the learning strategies practised in the Y1 *TM* lessons, would now receive further *TM* lessons at a rate of about one every 10 days. In addition the teachers would be encouraged, where possible, to use the

Figure 6: Proportion of children per class at the Mature Concrete level at Post-test on Spatial Relations



same teaching skills within the context of their ordinary Numeracy work, and also to establish a link from the children's *TM* insights and the National Curriculum learning objectives. The research design involved using a Piagetian Spatial test early in Y1, and then again at the end of Y2, and also Key Stage 1 results on Maths. Figure 6 shows the effects on the Piagetian test at the end of Y2

The Piagetian Spatial test from the CSMS survey is answered only in terms of the children's drawings. It was modified for 5 year-olds as a Pre-test by asking them only to complete figures, e.g. for water in a jar, the jar was drawn for them: they had only to pencil in where the water was. But for a post-test, after trialling with 7 year-olds, the Spatial test was found usable entirely in terms of the children's own drawings, as previously with 10 year-olds. Although it is cast in the spatial mode, the test in fact acts as a test of fluid intelligence: one assesses, in each drawing made, the number of relations the child was bearing in mind in the act of drawing. In drawing trees on the side of a hill, if they consider one relation (Early Concrete) then they draw them perpendicular to the side: if they can manage two relations (Mature Concrete) then they draw them perpendicular to earth.

As a guide to the Piagetian scale used in Figure 6, '3' is the Early Concrete level (2A), '4' is Middle Concrete (2A/B) and '5' is Mature Concrete (2B).

In the monograph Shayer, Demetriou & Pervez, 1988 surveying 5 to 10 year-olds, it was shown that the proportion of 7 year-old children in Greece, Australia and Pakistan succeeding on a least two-thirds of Piagetian tests at the 2B level was between 15 and 20%. This is similar to the proportion shown in the CSMS survey for 14 year-olds at the Early Formal level ('7' on the Piagetian scale) and also the proportion selected for grammar school education at 11^{+} in the 1945 Act. It seems reasonable to assume that it is those children, at 7 years already at the 2B level, who are the future high achievers who would have been selected at 11 for selective education. In Figure 6 it can be seen that all but two of the experimental classes were well below the National average at Pre-test in Y1, yet six of them had over 40% at the 2B level at the end of Y2. Mean effect-sizes were Spatial test, 0.63 S.D. and Key Stage 1 Maths, 0.38 S.D. The expectation is that this will affect the children's learning experience during the rest of their time in Primary school.

DISCUSSION

Enough evidence has been presented to show that the research has engendered class management skills in the teachers involved that realise Vygotsky's insistence that teaching should foster development as well as subject knowledge: that it should always aim ahead of where students presently are. Yet in achieving this we have had to abandon almost completely the cause-and-effect thinking present in so many Government initiatives. Teachers mediate the collaborative learning through which their pupils mediate each other. For them to gain possession of the underlying theory—both social and cognitive— they need the same mediation of their own collaborative learning as we are asking them to use with their children. This places the University mediator in an interesting and demanding position: more akin to a sports coach than a knowledge 'expert'. The essential difference between this methodology and all others is the view that both pupils and their learning objectives can be interpreted, as with Rasch analysis, on one and the same Piagetian scale of difficulty. For the theory and practice of maths teaching itself, one can imagine ahead, in the spectrum of teachers' skills a seamless integration of instructional teaching—aimed at increasing children's competence in what they already understand—and interventionist teaching, aimed at enhancing children's cognitive development. This process, only the first word, now needs further development in other people's hands: the authors are only too aware of how much more there is to understand and make explicit than they have hitherto succeeded in expressing.

NOTES

1 *Cognitive Acceleration in Mathematics Education I* (1993-1995) project funded by the Leverhulme Foundation. *Cognitive Acceleration in Mathematics Education II* (1995-1997) project funded jointly by the Economic and Social Research Council and the Esmée Fairbairn Trust.
2 *Realising the Cognitive Potential of Children 5 to 7 with a Mathematics Focus* (2001-2004). Project funded by the ESRC at King's College.
3 *Concepts in Secondary Mathematics and Science* (1974-1979). Research Programme funded at Chelsea College by the Social Science Research Council
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WHY WE ARE STILL TEACHING: A SMALL-SCALE STUDY OF MATHEMATICS TEACHER RETENTION

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This paper reports on a small-scale study of ex-students from a PGCE course and the progress of their teaching careers. Contact was made with ex-students from seven cohorts, going back as far as 1976 and respondents were asked to complete a questionnaire. It explored their reasons for either remaining in teaching (e.g. working with children) or leaving (e.g. workload). This paper suggests areas in which progress is being made in retention (e.g. pay and conditions).

INTRODUCTION

For several years the TDA (previously TTA) has been addressing the related issues of *teacher recruitment* and *teacher retention.* In 2002 we decided to explore the issue of retention, i.e. teachers' decisions about staying in the teaching profession. We developed this study further in 2005.

Teacher retention has been an issue in a number of countries for many years. The NFER teacher survey, undertaken for the GTCE, identifies key factors which teachers enjoy about teaching, the main one being the satisfaction of helping children academically and personally (NFER, 2005). They also report the frustrations:

> 'The main frustrations of teaching, as reported by teachers, are insufficient time to plan and prepare, a lack of work/life balance, the amount of paperwork and the poor behaviour of some pupils.' (NFER, 2005)

The issues of working conditions and teacher salaries are often cited as negative factors (Lambert, 2006). In Australia, 8 stages of a teacher's career were identified, including the movement from 'Career entry: reality shock – survival and discovery' to 'Stabilisation: developing commitment' (Schools Council, 1990, quoted in Manuel, 2003, p. 143).

'Stayers, leavers, lovers, and dreamers' an editorial by Marilyn Cochrane-Smith in the Journal of Teacher Education (Cochrane-Smith, 2004) reviews several studies of teacher retention in the USA. The title of the piece reflects some of the reasons teachers stay: their 'love' for the students they work with and their 'dreams' of making a difference in students' lives. One of these studies also highlighted reasons teachers leave, including 'low salaries, student discipline problems, lack of support and little opportunity to participate in decision making' (Ingersoll, cited in Cochrane-Smith, 2004, p. 388)

METHODOLOGY

The study has been undertaken in two parts. The first stage was an investigation of three cohorts of ex-students, from 1999, 1996 and 1991. We surveyed all the ex-Bristol PGCE mathematics students whom we could track down from these groups. We received 17 responses to our questionnaire and then we also interviewed two, both of whom were heads of department in local schools. The findings from this part of the project were reported at an Open University conference on teacher recruitment and retention in January 2003 (Brown and Winter, 2003). They largely reflected those reported on the previous page in the NFER survey. We then decided to extend the survey to earlier groups and so contacted ex-students from a further four cohorts, from 1987, 1983, 1980 and 1976, as well as obtaining additional responses from the first three cohorts. Altogether we received responses from a total of 48 ex-students. Of these nine had left teaching. This should not be taken as statistically meaningful in that it was obviously easier to find ex-students who are still teaching. Our aim was to take a positive view of the issue – why do teachers **stay** in teaching, rather than why do they leave.

We extracted comments from the questionnaires, categorising them into broad areas. This was largely qualitative data and the data set was not of a sufficient size for statistical examination, but clear themes emerged from our examination of the responses. We recognise that others, from their past experiences, would have identified different categories. For instance, one participant at the BSRLM day conference, began by looking for instances of the word 'mathematics' and found none. Some of these themes changed between the first and second phase responses and these points are outlined below.

SOME KEY POINTS FROM THE DATA

All questionnaires:

<table>
<tr><td>Good things about teaching:

•1 Aspects of working with children, and the rewards, were key for most respondents – including those who had left teaching. This was the most frequently mentioned factor for both groups.

•2 Intrinsic interest and variety in the job was important for those who stayed – but not mentioned by any leavers.

•3 Intellectual challenge was important for those who stayed – but mentioned by only one leaver.

•4 Working with interesting and supportive colleagues was mentioned by about a third of all respondents – an equal</td><td>Bad things about teaching:

This was a much more diverse set of responses with few key features common to large numbers. Only three were mentioned by a reasonable number of respondents:

1. Unmotivated pupils or poor behaviour – mentioned by more than half of leavers and more than a quarter of stayers.

2. The amount of paperwork was mentioned by about a third of stayers and 2 (a quarter) of leavers.

3. Workload out of school hours was mentioned by more than half of leavers, but only a sixth of stayers.

Other factors were all smaller scale than these three, but the following were mentioned by a</td></tr>
</table>

proportion from both groups. •5 The enjoyment of seeing pupils understand new ideas was important to both groups – and was mentioned by a higher percentage of leavers than stayers. •6 Good holidays were important for those who stayed (although many added that they needed them) but were not mentioned by leavers. •7 More than half of leavers said they had enjoyed planning lessons – no stayers mentioned this.	few: • Lack of resources, • Teaching and managing coursework, • 'Irrelevant' tasks, • Ofsted and associated stress, • Unsupportive parents, • National exams.

Responses to the 'bad things' question (see above) were much more at a level of individual detail, so more difficult to categorise. They seemed to reflect individuals' circumstances more than 'the bigger picture'. One third of original questionnaire respondents mentioned inadequate time for preparation and development of teaching – but this was only mentioned by one 'second round' stayer. Perhaps this has improved? Similarly, more 'first round' respondents mentioned pay/status. Again, this may have improved.

Future career plans – five years time:

1 Most teachers felt they would still be teaching, either in the same job or promoted.
2 About a sixth thought they might not be teaching, some combining this with plans for childcare.
3 Of leavers, a third (3) thought they might return to teaching.

About two thirds of stayers said they had considered leaving teaching at some point, for a range of reasons including the pressure of the job (but this was reduced in the second round responses), feeling undervalued, frustration with 'the system' and excessive workload. Their reasons for staying included enjoying it on balance, needing the salary, being refreshed after a break or taking up new opportunities and not yet having made the move.

Since the study was conducted, one of the 'leavers' has actually returned to teaching, in a senior post, and has commented to us that his experience outside teaching (working in educational statistics) has enabled him to bring a new perspective to his work in school.

Questions specific to leavers:

Reasons for leaving: •1 More than half mentioned a specific interest they wanted to follow up. •2 Others mentioned family reasons,	Links between teaching and current job: •1 Almost all had moved into a job closely related to teaching, or still in an educational context.

connected with time pressures. •3 Some progressed into related jobs. Only three mentioned any regrets about leaving teaching.	•2 About half still do some teaching in a different setting. Others mentioned some generic skills which have continued to be useful – organisation, communication, hard work and 'people skills'.

Second round questionnaires only:

In these questionnaires we asked a few extra questions, including the following:

Most valued professional development: •1 The most common response was to mention a specific training event on a particular topic. •2 Second to this was sharing experiences with colleagues •3 Working with PGCE students and/or presenting training themselves was mentioned by about a quarter. •4 Other popular activities included school run session with colleagues, mutual lesson observations and reflection on one's own teaching.
Advice to a graduate considering teaching: •1 The most popular response was some variation on 'try it – it's rewarding'. This was from both stayers and leavers. •2 Others tempered this with a warning to be prepared for hard work – this was more prevalent among leavers. •3 Some advised that it would get easier over time – this was more common from stayers. •4 Leavers often advised exploring teaching first by observations in school. •5 Leavers also advised taking advantage of the opportunities teaching offers, such as teaching overseas or extra-curricular activities with pupils.

DISCUSSION

We called our paper reporting on the first study 'Beyond the breaking waves' as it seemed to us that one factor was that of getting enough experience 'under one's belt' to have the confidence to make one's own decisions about how to prioritise parts of the job rather than be constantly 'hit' by new demands. This seemed to take about five years – if teachers stayed that long, they often stayed for many more years.

Now that we have extended the study into earlier years, we continue to see this effect. As teachers gain experience they, unsurprisingly, become more confident to set their own agendas for their work. They also become more 'hooked into' the work through dependence on their salaries and through their personal commitments (families and mortgages) so that they are less flexible and cannot afford career changes and the accompanying salary drops. These two factors mean that those who have made it 'beyond the breaking waves' are more prepared to accept and work on the factors they don't like about teaching, while staying in the job.

Poor pupil behaviour and motivation, cited as an issue by Manuel (2003, p147) for teachers in the early parts of their careers and mentioned in the NFER teacher survey (NFER, 2005), was one of the important factors for our respondents. The point made by Manuel about the importance of support during this key time, when teachers could be moving into the 'Commitment' phase of their careers, was not specifically mentioned by our respondents. Manuel's study mirrors ours in many of the positive factors identified by teachers who stay including: effective pre-service education, participation in collaborative groups such as support networks and professional associations, opportunities for promotion, financial need and a 'gritty determination' to see though one's ideals (Manuel, 2003, p148)

Intellectual challenge was mentioned by a greater number of respondents in the second phase of the survey, perhaps indicating that this is a factor which increases with experience (as this phase took views from those who had been in teaching for longer) and with confidence in one's own practice.

An important issue emerging from the two phases of the study is that there do seem to be changes in teachers' experiences in recent years which they see as improvements to the job and therefore as encouragements to stay. The recent workload initiatives of the government do seem to be impacting on teachers' experiences positively. Two areas in which improvements seemed to have occurred are time for preparation and planning and teachers' pay/status. Both are areas which have received government attention in recent years which is perhaps proving successful. The vacancy rate for secondary mathematics teachers, down from 2.1% in 2001 to 1.0% in 2006 [1] (DfES, 2006) would seem to reflect this improvement, although clearly there are many factors underlying these figures which would need deeper examination. However, it is interesting to note that the NFER teacher survey does not yet seem to reflect these improvements (NFER, 2005).

The fact that many of the 'leavers' had not in fact left education is an important finding. Most were still engaged in related work, including one in Teacher Education (Jan Winter!). Indeed, a US study suggests:

> '…we may need an expanded notion of retention that recognises the migration to leadership roles not as failure to retain but as an appropriate career path for some…'(Teacher Education Program Research Group, UCLA cited by Cochrane-Smith, 2004, p390)

The teachers surveyed were all ones who had received their Initial Teacher Education via the University of Bristol PGCE course, and therefore knew the institution and in some cases, the researchers. It may therefore be that there were some 'local issues' regarding the particular training they received. As Stokking *et al*. comment, a 'thorough and realistic preparation' for teaching may help reduce negative impacts of 'practice shock' so that teachers are more likely to stay through their early years of teaching (Stokking *et al.*, 2003, p331). Some respondents' comments indicated they felt they had experienced this. It may also be that their responses were phrased more positively because of their previous, and in some cases ongoing, relationships with us.

It is important to reassert that this study is small scale and that the findings may not be statistically significant on the large scale. However, it does provide some evidence that teachers of mathematics do enjoy their work and that those who stay have strategies for 'maximising the positive'.

NOTES

1. Vacancies as a percentage of teachers in post. In both years quoted, the vacancy rate for teachers of mathematics was about 1/3 higher than the overall vacancy rate. The figure for 2001 was a high point in the data, with vacancy rates from 1997 building to 2.1% in 2001 and decreasing since then.

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INFORMING THE PEDAGOGY FOR GEOMETRY: LEARNING FROM TEACHING APPROACHES IN CHINA AND JAPAN

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An authoritative report into the teaching and learning of geometry argued, amongst other things, that the most significant contribution to improvements in geometry teaching are to be made by the development of good models of pedagogy, supported by carefully designed activities and resources. This meeting of the Geometry Working Group provided an opportunity to consider approaches to the teaching of geometry developed in China and Japan and to review what research might have to contribute to developing new pedagogic approaches.

INTRODUCTION

Geometry is recognised as not only one of the most important components of the school mathematics curriculum but also, alongside algebra, as one of most important elements of mathematics itself (Royal Society, 2001; Atiyah, 2001). The reasons for including geometry in the school mathematics curriculum are myriad and encompass providing opportunities for learners not only to develop spatial awareness, geometrical intuition and the ability to visualise, but also to develop knowledge and understanding of, and the ability to use, geometrical properties and theorems (Jones, 2000, 2001, 2002). This being the case, it can be argued that teaching approaches need to encompass the encouragement of the development and use of conjecturing, deductive reasoning and proof, as well as developing skills of applying geometry through modelling and problem solving in a range of contexts (including real world ones), and an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry (Clausen-May *et al*, 2000; Jones, 2000). All these considerations tend to make geometry a demanding element of mathematics to teach well, especially when other topics in the mathematics curriculum (such as numeracy and algebra) can dominate curricula considerations (Jones & Mooney, 2003).

In terms of effective pedagogy for geometry, the general situation appears to be that despite many efforts, as Howson (2003) attests, “Euclid-style geometry [is] found extremely difficult (and often uninteresting) by most [school] students”. Nevertheless, in a number of countries there are teachers continuing to work hard at designing lessons that focus on helping learners make the difficult transition to deductive thinking in geometry. This report focuses on how improvements in geometry teaching might result from the development of good models of pedagogy, supported by carefully designed activities and resources, and how a consideration of approaches to the teaching of geometry developed in China and Japan might inform research designed to contribute to developing new pedagogic approaches for geometry.

TEACHING GEOMETRY IN CHINA AND JAPAN

The opportunity to focus on China and Japan is useful as there are interesting similarities and contrasts. For example, both countries have National Curricula for mathematics that cover geometry, amongst other mathematical topics (JSME, 2000; Ministry of Education PRC, 2001). Yet, for teachers in the two countries there are different traditions and different ways in which they have responded to international developments over the years (Jones, Fujita & Ding, 2005).

In this paper we provide examples of geometry teaching designed for Year 9 (called Grade 8 in both China and Japan) from each of these countries.

Teaching Year 9 geometry in China: a theorem about parallelograms

In the following extract from the lesson, T is the teacher, Ss represents more than two students, other letters are individual students, names are pseudonyms.

T: So far we have only known one way to prove a shape is a parallelogram, that is, to use its definition to prove. This means that we need to prove that two pairs of opposite sides are parallel. So, here, we use '{' to show that we need to prove that both pairs of opposite sides are parallel. (The teacher wrote down '{//, //' on the blackboard).

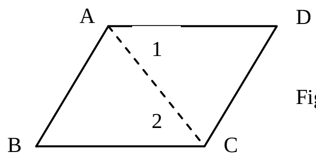

Fig. 1

T: To prove two lines are parallel, what methods did we learn early in grade 7?

Ss: The alternate interior angles.

T: We could use the 'three lines and eight angles'*, couldn't we? (*three lines refers to when two parallel lines are intersected by a transversal; eight angles mean that eight angles, less than 180˚, are formed by the three lines). So we need to create the basic figure of three lines and eight angles here. I need to link points A and C. Of course, you could link B and D (The teacher linked A and C in figure 1).

T: Now, I linked A and C. If I want to prove AD//BC, what actually must I prove here?

Ss: Angles are equal (means angle 1 and angle 2).

T: If I want to prove angle 1 is equal to angle 2, what should I prove? (The teacher wrote down // ➔ 1= 2 on the blackboard).

Ss: Congruent triangles.

T: Good. Now we need to prove congruent triangles. Considering about the condition for proving congruent triangles. Here, we have already known that two pairs of sides of triangles are equal. Moreover, the two triangles (means triangles ABC and ADC) share one side. So it is quite easy to prove the congruent triangles, isn't it? (The teacher wrote down 1= 2 ➔ (sss) on the blackboard).

T: So, from now on, as we have learned geometry for over one year, we need to know how to analyze the problem, when we prove a fact. A good student always considers the methods used for proving the problem. An average student still tries to learn how to write logical paragraphs. Here, you should firstly write the proof about congruent triangles. Secondly, you need to write the proof of equal angles. Finally, you write the proof of parallel lines. OK?

T: Now, you know how to prove one pair of opposite sides of the quadrilateral are parallel. In the same way, you could prove another pair of opposite sides of this quadrilateral. But you do not need to write the proof again, as the method is as same as that of the first. You could omit one logical paragraph, and directly provide the fact. Therefore, the quadrilateral is a parallelogram. What is the reason?

Ss: (Very low voice). The definition of parallelogram.

T: Now the method to prove this problem is clear. In the previous lesson, Ning did not fully prove this problem due to the limited time. I pointed out a disadvantage of her proof writing. I still emphasize here that you should write correspondingly the vertices of congruent triangles. In this figure, you need to write that triangle ABC is congruent to triangle CDA. As in triangle ABC, A is opposite to BC.

T: OK, we know that the converse statement of the property of a parallelogram (the property is that 'both pairs of opposite sides of a parallelogram are parallel) is a true statement. This converse statement is that 'if two pairs of opposite sides of a quadrilateral are equal, then this quadrilateral is a parallelogram'. Is this converse statement about the property of a parallelogram? Or about verifying a parallelogram? You see, the conclusion is a parallelogram. So it is used to verify whether a quadrilateral is a parallelogram or not. Therefore, we use this statement as a new theorem to verify parallelogram.

T: So far, except for the definition of parallelogram, we learn a new theorem to verify parallelogram. (The teacher wrote down the lesson title 'The theorem of verifying parallelogram' on the blackboard). Well, who could use words to state this new theorem again? And who could use mathematical language to state the theorem? Lin, could you?

L: If two pairs of opposite sides of a quadrilateral are equal, then this quadrilateral is a parallelogram.

T: Very good. (The teacher repeated the theorem). We could also use mathematical language to state this theorem. In quadrilateral ABCD, what is already known? See, we know that tow pairs of opposite sides are equal. AD=BC, and AB=CD. So I could tell that this is a parallelogram. The reason is the theorem we just learned. (The teacher repeated the theorem). (The teacher wrote down 'In quadrilateral ABCD, AD=BC, AB=CD ( ), Quadrilateral ABCD is a parallelogram. ( )' on the blackboard). We need to use words to state the theorem after we verify that this is a parallelogram. So what is it, if two pairs of opposite sides of a quadrilateral are equal?

Ss: It is a parallelogram.

Teaching Year 9 geometry in Japan

The way teachers structure their lessons in Japan is influenced (as in China and, no doubt, elsewhere) by the specification of the mathematics curriculum, the demands of examinations, and the design of textbooks. Our analysis also suggests that lesson designs in Japan are also influenced by the occurrence of 'Lesson Studies' and by recent Japanese research into the learning and teaching of mathematics. For example, 'Lesson study', practiced by teachers in Japan for the last several decades, is one of the most common forms of professional development and involves teachers working in small teams collaboratively crafting lesson plans through a cycle of planning, teaching and reviewing (Yoshida 1999). Through this process, Japanese teachers appear to have collaboratively developed a view about 'good lessons of mathematics'.

For example, to teach the properties of the parallel lines and ratio in Year 9, teachers would organise a lesson (50 min.) as follows. First, a problem for the day is introduced. To understand the relationship between the parallel lines and ratio, it is useful to know properties of similar triangles. Thus, this lesson could start from a problem 'Let us prove that if PQ//BC in a triangle ABC, then triangles APQ and ABC are similar to each other'. Then, in the development stage, students would undertake to prove this problem, either individually or in groups. Proof of this problem is shared in a whole classroom, and finally, the topic of this lesson is summarised as 'If PQ//BC in a triangle ABC, then triangles APQ and ABC are similar to each other, and therefore AP:AB=AQ:AC=PQ:BC, and if PQ//BC then AP:PB=AQ:QC' (Summary stage).

Kunimune *et al* (2002, p. 69) state that the lesson plan above is a typical one in Japanese secondary schools (we call this 'Typical approach') which is often a result of the lesson study, and could be time efficient, but they speculate that students might be rather passive in this format. As an alternative approach, Kunimune et al (ibid., p. 69-70) propose that a lesson can start from geometrical construction (i.e. construction by only ruler and compass), and we call this approach 'Construction approach'. This lesson starts from a more challenging construction problem 'Let us consider how we can trisect a given straight line AB.' Then, one of ideas from students will be chosen and proof of it will be considered within group work. Finally a theorem is introduced and summarised as 'In a triangle ABC, P and Q are on the line AB and AC respectively. If PQ//BC then AP:AB=AQ:AC=PQ:BC and AP:PB=AQ:QC.' which students would have noticed during the construction activities.

This approach would make the lesson more active, and encourage students to consider why constructions work. Also, students would be able to discover theorems/properties of geometrical figures through construction activities. In fact, the report by the Royal Society/Joint Mathematical Council working group suggest that "the mathematics curriculum should be developed to encourage student to work investigatively, demonstrate creativity and make discoveries in geometrical contexts so that students develop their powers of spatial thinking, visualisation and geometrical reasoning" (Royal Society, 2001, p. 10). A possible disadvantage is, however, that this lesson could be very time consuming in this format (the suggested plan is 3 hours).

It is difficult to conclude 'Construction approach' is better than 'Typical approach' or vice versa, and rather a task for us is to consider what research would be necessary to establish a good pedagogical model in the teaching of geometry.

CONCLUDING COMMENTS

In both China and Japan, the teaching and learning of deductive reasoning in geometry remains a major objective in Year 9 (Grade 8) but this is not without its problem. For example, research in Japan indicates that while most 14-15 year-old Japanese students can write down a proof, around 70% cannot understand why proofs are needed (Kunimune, 2000). Similar results with a UK student who was educated in Hong Kong are reported in Healy and Hoyles (1998; p. 166).

To counter this, teachers in both countries not only try to maintain students' interest, but also aim to assist students in creating definitions and conjectures and in gaining insight into new geometrical relationships and inter-relationships.

The recent UK study of geometry teaching (Royal Society, 2001) concludes that "the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources" (p19). The analysis of lessons given by mathematics teachers in countries such as China and Japan might inform the development of new pedagogical approaches to teaching geometry. In future research it is necessary to consider questions such as 'What pedagogical background is underpinned in Chinese and Japanese geometry teaching?', 'What can we learn from Chinese and Japanese geometry teaching?', 'What theoretical and methodological models should we develop to examine effective geometry lessons?' etc.

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ISSN 1463-6840