# PROOF, REASON, ABSTRACTION AND LEAPS: A CULTURAL-HISTORICAL APPROACH TO TEACHING GEOMETRY

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*Geometry for most learners is 'shape and space' without reason, deduction and proof. There seems to be the assumption that deductive geometry is inappropriate, either because it is difficult to learn or that there are no obvious benefits. We take the view that most secondary school learners are capable of engaging with the abstract and rule-governed intellectual processes that became the world's first fully developed and comprehensive formalised system of thought. This article discusses a curriculum initiative that aims to 'bring to life' the major 'transformative events' in the history of Greek geometry, aims to encourage a meta-discourse that can develop a reflective consciousness and aims to provide an opportunity for the induction into the formalities of proof and to engage with the abstract.*

## DISCUSSION ARTICLE

Over the years we have asked many UK mathematics educators and teachers how they would introduce the angle property of the triangle. Always the response has been 'by getting the class to measure the angles'. One limitation of this approach is that it is unlikely for students competent in using a protractor to measure $180^0$ exactly. If a student's experience suggests $179^0$ then who is the teacher to state otherwise? There is, moreover, a fundamental sense in which this approach is limited: it undermines the capabilities of most students to prove the angle property. Experience has shown that low 'ability' students are capable of abstract reasoning regarding proof. For example, in a recent classroom observation of a trainee placement and a year 7 class of low 'ability', one student complained that the tearing of the angles of the triangle demonstration to show the angle property only applies to the triangle torn in the demonstration. Of central importance is the fact that 'proving' by empirical or intuitive means engages fundamentally different cognitive processes than does proving by means of logic. The use of protractors may help to support the conclusions of logical analysis by marshalling an empirical demonstration, but it secures a very different form of conviction and understanding, and it misleads the student with respect to the pedagogical goals of advanced mathematical study.

Geometry in secondary education has become primarily Shape and Space with little or no deductive geometry. This, we argue, constitutes in part an educational disenfranchisement because it denies learners the opportunity to be inducted into a formalized cultural system. In the attempt to motivate students, proof, rigour and formalism have been downplayed. This may sound journalistic but it is fair to say that the curriculum has been dumbed-down with the assumption that tying it to everyday phenomena is the key to motivation. Contrary to what was expected, this dumbing-down has contributed to the disrespect students feel toward a system that legally obligates their time yet gives nothing in return except qualifications valued by those who feel they have a stake in the system. Although the majority of students are not going to become mathematicians, scientists, engineers or economists, nevertheless, deductive geometry has an intrinsic value because it can develop, even amongst the most concrete thinkers, the ability to think logically and in the abstract. This is the disciplinary (or developmental) argument for the teaching of deductive geometry in schools: the development of the potential to logically reason and to think in the abstract. The disciplinary argument naturally leads onto the cultural-historical argument for teaching deductive geometry: the development of logical and abstract thought essential for *full* participation in the technological world that surrounds us.

Why geometry? Geometry has an historical significance because of its impact in transforming human culture. It initiated and sustained a 300 year conversation in which rule-governed cognition, abstraction and formalisation were developed. This conversation became integrated into the host culture, as evident in the art and architecture of the Classical period (Kline 1982). Without this conversation, modern technological society would not have been possible. To highlight this historical significance is not to suggest any recapitulation theory, but education can be seen as an induction into the use of those cultural tools that significantly transformed consciousness to which the history of a subject can help identify (Egan 1997, 26-31). Those cultural tools can be introduced through a narrative historical treatment that can provide a human context that is easier to identify compared with introductions that are totally abstract and symbolic and, hence, unnecessarily dogmatic.

Geometry provides an ideal venue for inducting students into proof and the formalism of mathematics and to encourage them to *think* as mathematicians. Geometry is ideal for a meta-discourse that can develop reflective consciousness about ones own reasoning processes independent of prior experience. By immersing students into this conversation they can become involved in a meta-discourse concerning the objects of discourse and become engaged in reflection and abstraction – requisites for thinking as mathematicians and as aids to thinking within a formalised system.

What we propose is the introduction of seventeen or so 'primary events' (i.e. key ideas) in geometry's developmental history (Carson and Rowlands, 2005). The amount of time required to address these primary events in the classroom is manageable, so we are not suggesting an overhaul of the curriculum, unless of course teachers decide to exploit these primary events into a full-blown deductive geometry course of formalism and proof. The point is that each primary event can establish a bounded region of cognitive activity. These bounded regions can be explored through well-designed activities, exercises and discussions. Once a primary event has been conceptually unpacked then the class can move onto the next primary event to explore an enhanced level of rigour, formalisation and abstraction.

From 1871 to the mid Twentieth Century, Euclid's Elements was taught so as to develop logical reasoning. Although the value of teaching classical Greek geometry was recognised, Euclid's Elements was not a suitable secondary school text-book and it was not designed as such. The rote learning of the Elements was not conducive in developing an intuition for proof, and during this period attempts were made to combine practical with theoretical geometry so as to develop this intuition. For example, the Mathematical Association's three stage schema whereby the first stage is practical, the last stage theoretical and the transitional middle stage mixing both (M.A. 1923; M.A. 1939). The mixing of practical with theoretical geometry may not have had the success that was anticipated because it was not clear how the practical can develop logical intuition. Although the practical has a part to play in understanding the theoretical, by merely mixing both the former can also impede the latter in the sense of students habitually focussing on how diagrams appear, rather than thinking about what is actually *given* in the diagram. For example, two lines that *appear* parallel taken to be parallel or an angle that *appears* to be bisected by a constructed line taken to be bisected. Playing with the practical does not necessarily secure progress towards logical intuition. What is required, we argue, is an explicit meta-narrative that can guide students through the key process of abstraction and secure an understanding of proof as both a logical and socially constructed process.

The origin of Euclid's Elements did not consist in an exercise of definitions, theorems and proofs, but nearly four centuries earlier in the practical geometry of the Egyptian priests, which mainly consisted of areas and volumes obtained by trial and error. Although their practical geometry did involve abstraction, such as a rectangle to represent a field, they did not have the idea of *the* rectangle as an abstract geometrical object. For them, the rectangle was still a fence bounding a field. If you were to ask an Egyptian geometer 'what is a straight line?' then the geometer would most likely hold out a stretched rope and say '*that* is a straight line!' and although they had a concept of a straight line, this concept was of some concrete exemplar. By visiting the Egyptians and studying their geometry, Thales introduced, for the first time in history, a method of finding the *relations* between the different parts of a figure. That method was reasoning and made possible because of abstraction, that is, the creation of *ideal* geometrical objects divorced from any concrete exemplars.

Thales may have witnessed the priests' use of wooden stakes and stretched rope that formed configurations such as a line segment, a pair of intersecting lines or a 'Pythagorean triple'. By his very efforts in demonstrating something would be true for all cases of a given geometrical configuration, there must have appeared the notion of necessary truth and the notion of ideas existing above and beyond those concrete exemplars. *This* is the origin of Euclid's Elements: *the combination of the intellectual act of abstraction and the possibility of formalised, logical proof*.

These two transformational or primary events in geometry's developmental history can be utilised to develop logical intuition and the ability to think in the abstract. To guide students through the process of abstraction, 12 or 13 year olds are taken outside onto the playing field where four stakes are hammered into the ground and two intersecting ropes form an X. The students are asked to draw *exactly* what they see. Some students draw an X but they are encouraged to draw what they actually see.


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The students are then asked to consider the relationship between *the actual* configuration in the field and the *literal representation* drawn. The teacher points out that this is an act of *abstraction* and asks them to consider what was carried forward from the actual configuration and what was left behind. They are invited to discuss the matter amongst themselves and some offer examples of what was left behind such as details of landscape, the material of the rope or the shape of the wooden stakes. The students are then shown a model made from a wooden board about the size of a textbook with four wooden pegs and two stretched pieces of string forming an X. The teacher states that this is a *model* of the actual configuration and asks how it is similar and how it is dissimilar to the actual. Some students give details of the abstraction but some also hint at the model giving the opportunity to view according to desired perspective (e.g. 'you can rotate the model'), so already there is an object of contemplation to discuss at this level of abstraction. Next the teacher asks the students to make another drawing, replacing the sketch of each wooden stake with a dot and the sketch of each rope with a straight line. The students are asked to consider and discuss what is different, what is similar, what has been carried forward and what has been left behind. Consolidating the discussion, the teacher explains that this is a more *abstract* drawing than the *literal* drawing and that the configuration has been drawn fourth while the materials used in the actual configuration have been left further behind.

The students are now asked to move from the drawings to their imagination and to consider their idea of the configuration they have been working with. They are asked to close their eyes and hold in their 'mind's eye' the image of the two lines intersecting. They are told that this is 'an idea of two straight lines intersecting' and that this idea is a concept that they hold in their mind's eye. Again they are invited to consider and discuss what has been carried forward and what has been left behind.

This is a crucial stage of abstraction and development: from the contemplation of particular configurations towards a universal concept of all such configurations. In imagination, these figures can be played with freely, adjusting lengths and angles at will. The concept serves as the direct object of discussion while the drawing is relegated to serve as an aid.

Next the teacher discusses the distinction between the concept as it resides in the student's mind (the *personal* concept) and the concept as it is represented in textbooks and understood by mathematicians (the *authorised* concept). The teacher explains that at this level of abstraction we have a hypothetical concept the meaning of which is agreed amongst and governed by the community of mathematicians. As a cue and a tease for the next and final level of abstraction, Plato's Realm of Forms, the teacher asks whether something is true by virtue of appearing in mathematics textbooks, or appears in the textbooks because it is true.

The teacher acts as if to downplay this next and final level by saying this is just for fun but has historical interest. However, this level is crucial in comprehending the terms of the discourse and forms a basis to appreciate the next transformative event, *proof*. The teacher explains that, for Plato, mathematics reveals certain truths that do not depend on humans for their existence and so therefore must depend on ideas that are independent of humans. Plato taught his students that their own ideas were imperfect reflections and fleeting glimpses of the Forms. He also taught his students that the Form of the circle, the line, the triangle and other geometrical objects were eternal templates more real than physical reality. Ontologically, this is an untenable doctrine, but it proved historically to be an extremely effective pedagogical heuristic [1]. For educational purposes it would be helpful to place the Forms in their socio-historical context: that *the Forms reflected an aspiration to understand the truth and not merely to accept what tradition says* [2]. This point is essential in understanding the modern cultural experience, so rather than dismiss this doctrine and utter 'Platonism' as a 'boo' word, we should teach it carefully.

Experience has shown that discussing the Forms can arouse the mind of even the most concrete thinkers. 'Gifted and talented' year 10 students are more than able to engage in philosophical discussion concerning the Forms (Rowlands and Davies, 2006). From 'widening participation' sessions plus school visits, experience has shown that year 9 'mixed ability' students can also engage meaningfully in such a conversation.

These higher levels of abstraction create an appropriate context in which to discuss and question the very objects of discourse that students are normally asked to work with in ordinary mathematics lessons, such as 'what is a geometric straight line?', 'what is a geometric point?' and 'what is an angle?' By asking such questions as 'what do we find when we have two geometric straight lines, three geometric straight lines, etc?' a whole new universe of discourse opens up by which the properties of the objects of discourse can be discovered. Perhaps more importantly, however, is that this level of abstraction is the stage to discuss proof as an exercise in reason and to enter the next transformative event: from contemplating two intersecting lines and knowing intuitively that all opposite angles are equal to actually learning how to prove it.

The significance is not immediately apparent and students may react that there is no need to prove something that is intuitively obvious. This is a rich historical moment, the very turn of imagination that caught Thales' mind. The teacher may follow Thales' course of thought and thereby state the question in a slightly different way that essentially asks: "If we constructed a demonstration that revealed the structure of this truth, what would such a demonstration look like?" In asking this question, Thales took the initial step in a process of cultural development that culminated in the Pythagoreans establishing a protocol for demonstrating proof. By engaging this cultural development students can join the Pythagoreans in their conversation and learn this protocol. Although the ancient Greek historians attributed proof to Thales, it is unlikely that he developed *formal* proof and perhaps went no further than cutting out templates to demonstrate opposite angles are equal. Pilot studies have shown that novice twelve year old students will generate the same kind of demonstration, which means they are on the right path to understanding how the method of proof developed. Some students may say that the very rotation of two intersecting lines proves the proposition that opposite angles are equal, but then they are already thinking as mathematicians. The point is to take them back to that historical moment and immerse them in the relevant problem space, guiding the transformation of their understanding as that primary event unfolds.

The aim of the project is to encourage thinking about thinking (meta-cognition, or a reflective consciousness) in the sense of the learner both reflecting on the objects of discourse and reflecting on her own thoughts in relation to those objects. A cultural-historical approach to geometry provides the relevant problem space with which to do this and can guide the learner from concrete considerations to the realm of pure abstraction, formalism and proof. As educators, wouldn't this be a fine thing to do?

## NOTES

1. Broadly speaking, it was a way to emphasise the distinction between concepts that are 'ideal', abstract and universal with concepts that are context-bound, concrete and unique. There was awareness that, for the first time in history, ideas become the object of thought. There was also the anxiety that the distinction could easily collapse, hence the invention of the Realm of Forms to help prevent that collapse. The Realm emphasised the existence of abstract concepts, formed for the first time in history since Thales. There is a sense in which abstract concepts do exist (e.g. Popper's World Three of the Objective Content of Thought), *where* they exist is problematic.

2. It could be argued that Plato's Forms represented the absolutism of a dying aristocratic order. Nevertheless, the sentiment of the Forms is similar to that of the Enlightenment: that truth is not dictated by person or tradition.

## REFERENCES


Carson, R. and Rowlands, S.: 2005, "Teaching the Conceptual Revolutions in Geometry". *Proceedings, Eighth International History, Philosophy, & Science Teaching Conference,* Leeds, England. July 15-18, 2005.

Egan, K.: 1997, *The Educated Mind: How Cultural Tools Shape our Understanding*. The University of Chicago Press, Chicago.

Kline, M.: 1982, *Mathematics in Western Culture*. Penguin, Harmondsworth.

M.A.: 1923, *The Teaching of Geometry in Schools*. Mathematics Association. G. Bell & Sons, London.

MA.: 1939, *A Second Report on the Teaching of Geometry in Schools*. Mathematics Association. G. Bell & Sons, London.

Rowlands, S. and Davies, A.: 2006, "Mathematics Masterclass: Is Mathematics Invented or Discovered?" *Mathematics in Schools,* July, Mathematics Association.