SCAFFOLDING, ABSTRACTION, AND EMERGENT GOALS

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This study examines the role of scaffolding in the achievement of a mathematical abstraction by focusing on emergent goals. An activity-theoretic approach to abstraction in context is taken. The examination is carried out with regard to the verbal protocols of two 17-year-old students working on a task related to the graphs of $y=f(|x|)$. This examination suggests that abstraction is likely to be achieved through satisfaction of several emergent goals. These goals are observed to be contingent upon four parameters: the scaffolder's interventions, students, tasks, and prior emergent goals. Dynamic and dialectical interrelationships amongst these parameters are discussed with regard to the students' verbal protocols.

INTRODUCTION

Abstraction is an important issue in mathematics education but there is no agreement upon a definition amongst educators (see Ohlsson and Regan, 2001; van Oers, 2001). This paper adopts an activity-theoretic model of abstraction (Hershkowitz, Schwarz and Dreyfus 2001), which is inspired by Davydov's (1990) epistemological theory adopting a dialectical connection between abstract and concrete. Hershkowitz et al., (ibid.) define abstraction as "an activity of vertically reorganising previously constructed mathematics into a new mathematical structure." The reorganisation of a new structure occurs through three epistemic actions: recognising, building-with, and constructing. Recognising means identifying a structure inherent in a given mathematical situation. Building-with consists of combining existing artefacts in order to meet a goal such as justifying a statement. Constructing consists of assembling knowledge artefacts to produce a new structure.

In empirical studies of abstraction researchers claim to gain insights into students' abstraction processes via the help of a knowledgeable agent e.g. the researcher/ interviewer. In these studies an interviewer supports students' work by free prompting (Hershkowitz et al. 2001), by providing the student with 'hinting' (Ohlsson and Regan, 2001) and by 'shifting the focus of activities' (van Oers, 2001). Thus the researchers seem to argue that abstraction is not an easy process which may be beyond the learners' unassisted efforts. This has links with the theoretical concept of 'scaffolding'.

Scaffolding may be defined as a guidance given to a learner to develop and achieve his/her fullest potential, which is beyond his/her actual present ability (Wood, Bruner and Ross, 1976). The key element of scaffolding is 'the sensitive, supportive intervention of a teacher in the progress of a learner who is actively involved in some specific task, but who is not quite able to manage the task alone" (Mercer, 1995, p.74). Scott (1997) characterises the idea of 'sensitive intervention' with three elements: monitoring present performance of the learner; analysing the nature of any differences between present performance and performance required by the learning goal; and assisting to respond with an appropriate intervention to address differences in performance.

Despite the implications of empirical investigations of abstractions for scaffolding, research studying the link between scaffolding and abstraction remains scarce. In an attempt to address this link, Ozmantar and Roper (2004) observe that supportive and sensitive interventions of a scaffolder to direct the students' actions and attentions (and regulate their work and effort) help students make progress towards abstraction. They argue that to do this, the scaffolder organises the main goal of the activity into subgoals and sub-subgoals. Yet, they do not characterise the emergence of subgoals and sub-subgoals in detail. The aim of this study is to characterise the emerging goals and relate this characterisation to the achievement of abstraction. For this purpose, verbal protocols of two 17-year-old girls (H&S) working together within a scaffolded situation are examined. The paper provides a brief description of the girls' joint work and the scaffolder's intervention and then discusses the interrelationships amongst scaffolding, mathematical abstraction, and emergent goal structures.

BACKGROUND

For the study, data were collected from the students working on four tasks connected with sketching the graphs of absolute values of linear functions. A diagnostic test was applied to select students based on two criteria: (1) they had the prerequisite knowledge needed to embark on the tasks; (2) they were not to be acquainted with the intended abstractions. 20 students were selected and organised so that 14 worked in pairs and six worked alone. Four pairs of students and three individuals worked within a scaffolded environment, the rest without.

This paper reports on protocols generated in the second task, by the end of which the students were expected to construct a method to sketch the graph of $y=f(|x|)$ by using the graph of $f(x)$, which is referred to as 'the structure of $f(|x|)$' in the paper. The second task was composed of five questions. The first question asked the students to draw the graph of $y=f(|x|)=|x|-4$ and to comment on any patterns in the graph. The second question asked if there were relationships between the graph of $y=f(|x|)=|x|-4$ and the graph of $f(x)=x-4$. In the third question, the graph of $y=f(x)=x+3$ was given and the students were asked if they could draw the graph of $y=f(|x|)=|x|+3$ by using the given graph as an aid. In the fourth question, four linear graphs without equations (referred to as 4A-4D in the paper) were given and the students were asked to obtain the graph of $y=f(|x|)$ for each one of these graphs. In the fifth question, students were asked to explain how to draw the graph of $y=f(|x|)$ by using the graph of $f(x)$.

PROTOCOL DATA

Some excerpts from H&S' protocol on the second task are presented in this section. H&S worked with the interviewer (I) scaffolding their work. Up until the 4th question, the students obtained the intended graphs accurately by substituting different values of x into the given equations. Later they moved on to the fourth question.

44I: Ok, let me remind you of that before this question you have been given the equation and graphs but here in this question you are just presented with the graphs without the equations. Are you planning to find the equation for each of the graphs?

45H: I think we should find the equation for the first given graph and later we can develop a general pattern to draw the others…

The students found the equation of the first given graph of $f(x)$ and then drew the graph of $f(|x|)$ by substituting. Having looked into the graphs of $f(|x|)$ that they have drawn so far, the students realised that the intended graphs could be obtained by 'taking some symmetries'. To do this, they drew a line passing through the intersection of $f(x)$ and the *y*-axis and then reflected a 'segment' of $f(x)$ in this line. Although they successfully applied this to draw the graph of $f(|x|)$ from the given graph of $f(x)$ in 4B, the students were not precise about which segment ($x \leq 0$ or $x \geq 0$) of $f(x)$ should be reflected. The scaffolder attempted to draw the students' attention to this lack of precision.

76I: I've got to ask a question: which part's symmetry is taken?

77H: In the line of y=2.

78S: According to the line… the line is passing when x=0

79H: According to this line [she refers to the line of y=2]

80I: Ok I understood according to which line you took the symmetry. This is not I was wondering. I want to know which part of f(x) is reflected?

81H: Oh, it is… we have problems to express it…

82I: Ok, you can say for example, the part on the right or left side of the y-axis.

83S: We take the symmetry [of the part of f(x)] on the left [of the y-axis] according to the line of y=2

The students then moved on to the graph of $f(x)$ given in 4C. They obtained an erroneous graph of $f(|x|)$ for the given graph of $f(x)$ in 4C in that instead of reflecting the segment of $f(x)$ at $x<0$, they reflected the segment of $f(x)$ at $x>0$. In 98, S realised that there was something wrong with this graph though she was unable to explain it.

98S: Before, we took the symmetry towards y-axis… but here we drew it… I don't know

99H: What do you mean?

100S: I mean, look, it should have been towards here not there

101H: It is not necessary… look, these two rays should be symmetric in the line of y=2

102S: Ok, look isn't this line of y=2? If we take the symmetry of that part in the line of y=2

103H: Which part?

104S: That part [the ray on the left side of the y-axis].

105H: No! We shouldn't take this part's symmetry

S's realisation that the graph was wrong gave rise to a lengthy argumentation (which continued until 116) in order to decide if this graph is right or wrong and the reason for this. Although S was right that the graph was wrong, she was unable to produce 'mathematically' convincing evidence. In addition, H seemed to be neither eager to accept S's explanations nor able to see the problem. As a result they got stuck and could not decide how to bring a resolution. Accordingly the scaffolder decided to assist them to tackle the problem (116) which brought about the construction of the structure of $f(|x|)$ as can be seen in the excerpt below.

116I: I'll suggest you something… I would like you to look at and examine the graphs that you have drawn so far… and discuss which part or parts always change, and which doesn't!

117S: [They return to the graph of f(|x|) obtained for the first question] look, in this graph… the graph of f(x) till the y-axis didn't change, and after y, it has changed…

118I: Do you mean that the part of f(x) at the positive values of x, which is always on the right of the y axis, doesn't change?

119S: Yes that's what I mean… and also on the right side of the y-axis, all of the values of x are positive… they are positive and so we don't change them…

120H: Positive values don't change?

121S: We are drawing the graph of f(|x|), right? I mean we are talking about these graphs…

122H: Yes, so?

123S: Umm… the absolute value sign is always outside of x, I mean it is |x|

124H: Positive values don't change in the absolute value sign…

125S: Exactly, it changes the negative values

126I: On the other hand, negative values of x, which are on the left side of the y-axis…

127S: They have to change… I mean in the absolute values sign, negative values change…

128H: Well, I mean there must be difference between the graph of f(x) and f(|x|) at the negative values of x

129I: Because?

130S: Because positive values of x remains unchanged in the absolute value sign, but negative values of x must be multiplied by minus to go out of the absolute values sign… so whatever changes occur in the graph of f(x), it must be at the negative values of x

131H: Ok, look let's put things together: as far as I understand, the part of f(x) on the right hand side (of y-axis) should remain the same, no matter what. And the part on the left hand side (of the y-axis) should change...

DISCUSSION

In the presented protocols there are three sorts of goals: the main goal of the activity, predetermined goals, and emergent goals. The main goal of the activity is the construction of the structure of $f(|x|)$ which is deliberately assigned by the researcher-cum-scaffolder. Fulfilment of the main goal is necessary to achieve the intended abstraction. However, the students need not necessarily be conscious of the main goal all the way through the activity to fulfil it. In the presented data, students gradually became aware of the main goal of the activity from the utterance 116I onwards. This awareness can be observed in the students' efforts to find a general method to draw the graph of $f(|x|)$ and thus to successfully construct the intended structure (see the utterances between 116 and 131).

The activity is essentially regulated by the task which is specifically designed to lead to abstraction. The task consists of purposeful questions each of which has their own goal(s) determined by the scaffolder. In this sense, the task can be considered as a pathway, among many other possible ones, constituted by predetermined goals which supposedly and potentially take students to a formation of the intended abstraction. However, the predetermined goal structure of the questions exist only in the 'head' of the scaffolder, they do not have to be necessarily seen and interpreted by the students in the same way as the scaffolder does. Thus, interpretation and perception of the questions and their predetermined goals result in an emergence of a new goal. For example, in the fourth question, four graphs of $f(x)$ without equations were presented and the predetermined goal was to urge the students to develop a general method to draw the graphs of $f(|x|)$ by using the given graphs of $f(x)$ rather than the equations. Despite the fact that the scaffolder tried to call attention to this (44I), the students interpreted this goal differently (45H) and decided to find the equation first. Consequently, a new goal different from predetermined one has emerged, that is, to draw the graph of $f(|x|)$ by substituting and then try to develop a general method.

Along with predetermined goals, the scaffolder also influences the emergence of a new goal. As the scaffolder has a clear vision of the target competence level and main goal of the activity, he can direct the students towards the achievement of the intended abstraction by setting new goals. To do this, the scaffolder monitors and analyses the students' performance on the basis of their actions and interactions and makes decisions about the type and amount of assistance; and this assistance may generate a new emergent goal. For example, when the students realised that they could draw a graph of $f(|x|)$ by reflecting, they were not aware which segment of $f(x)$ was being reflected. The scaffolder monitored this and, as a consequence of his analysis, intervened to assist the students (76I). The goal of the intervention was to have the students realise which segment of $f(x)$ was reflected. However, the students misinterpreted the intervention and talked about symmetry line. As a result, the scaffolder had to rephrase his question in 80I. Yet, the students failed to fulfil this goal because their explanations remained specific to the graph at hand. The scaffolder may also set an explicit goal, as a form of assistance, for the students to move forward. For example, when the students got the graph of $f(|x|)$ wrong in 4C and their argumentation was not fruitful enough to bring a resolution (see 98-105), the scaffolder in 116I set an explicit goal, that is, to examine the all graphs of $f(|x|)$ and decide which part has changed and which part remained the same. Hence emergence and fulfilment of a goal to a certain extent depend on how the students perceive and interpret the given assistance and relate it to the context of the activity.

Emergent goals are also dependent on how the students interact and influence each other and on how they perceive and assess their resulting work. For example, when they got the graph of $f(|x|)$ wrong for 4C, S's assessment about the graph's accuracy and H's perception of the resulting graph led to emergence of a new goal, that is, to evaluate and justify if the graph was right (or wrong). Both students were unable to produce mathematically valid arguments and counter-arguments, and this contributed to their failure to fulfil this emergent goal. In fact it was their unfruitful interactions and argumentations about the erroneous graph that prompted the scaffolder to set an explicit goal in 116I.

Finally, it should be pointed out that emergent goals are built upon each other in the sense that achievement or failure of a prior emergent goal gives rise to new emergent goal. Thus, prior emergent goals influence the character of the new emergent goals. For example, the students failed to satisfy the emergent goal between the utterances of 76I and 83S in that they could not make a general mathematical statement about which segment of $f(x)$ was to be reflected. As a result of this failure, the students also failed to obtain the accurate graph of $f(|x|)$ for 4C, which brought about the emergence of a new goal, that is, to look into the graph to check its accuracy. Subsequent to these emergent goals, a new goal set by the scaffolder in 116I emerged. The students fulfilled (though with the help of scaffolder, see 116, 118, 126, and 129) this emergent goal and consequently achieved the intended construction which required the students to assemble and reorganise the features of absolute value, of linear functions, and of symmetries in relation to the graphs of $f(|x|)$ (see 116-131). It is hence posited that goals continuously and successively emerge and that fulfilment of several of these emergent goals, though not necessarily all of them, may finally lead students to the formation of the intended abstraction.

The descriptions given so far should be considered as dynamic and dialectical rather than static interrelationships amongst these parameters i.e. emergent goals are the result of interplay amongst scaffolder, task, students and previously emergent goal(s). The critical point is that the participants continuously influence each other and the emergence of the new goals. That is, how the scaffolder perceives and interprets (monitoring and analysing) and then intervenes (assisting) and how the intervention is perceived and interpreted by the students, and how the scaffolder interprets the new situation and so on. It continues all the way through the formation of an abstraction and all of these interactions take place with regard to the tasks which create the context for involvement of the participants.

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