British Society for Research into Learning Mathematics

Proceedings of the Day Conference held at Institute of Education, London Saturday 9th November 1996

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We hope that members will use the proceedings to give feedback to the authors and that through discussion and debate we will develop an energetic and critical research community. We particularly welcome presentations and papers from new researchers.

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EXAMPLES, GENERALISA TION AND PROOF

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The interplay between generalisations and particular instances - examples - is an essential feature of mathematics teaching and learning. In this paper, we bring together our experiences of personal and classroom mathematics activity, and demonstrate that examples do not always fulfill their intended purpose (to point to generalisations). A distinction is drawn between 'empirical' and 'structural' generalisation, and the role of generic examples is discussed as a means of supporting the second of these qualities of generalisation.

IN[TRO]DUCTION

For all learners of mathematics there is the possibility of acquiring new knowledge by reflection on appropriate and relevant experience (and arguably there is no other way). Generalisation - unifying and information-extending insight - is central to such a means of coming-to-know, and may be viewed as a form of inductive reasoning. For the great mathematicians, as well as for novices, mathematics characteristically comes into being by inductive intuition, not by deduction.

> Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called induction. Newton was indebted to it for his theorems of the binomial and the principle of universal gravity. (Laplace, 1902, p. 176)
>
> I must admit that I am not in a position to give it a rigorous demonstration [...] The examples I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula. (Euler, translated by Polya, 1954, pp. 93-95)
>
> The purpose of rigour is to legitimate the conquests of the intuition. (Hadamard, quoted by Burn, 1982, p. 1)

The products of induction are plausible 'truth-estimates' (Rescher, 1980, p. 9), and such conjectures may well be held with conviction. But whereas initial regularity is so often a reliable guide to generality in mathematics, it is not invariably so. Consider the (false) propositions that n^2+n+4l is prime for all n (true for $n=l$ to 39), and that the number of regions ofa circle formed by joining each of n points (irregularly spaced) on the boundary to every other is a power of2 (true for $n=l$ to 5). In these two examples, the mere accumulation of confirming instances misleads. We shall argue that the quality of such empirical evidence is weak for mathematical generalisation, and indicate the need for other sources of conv~ion.

CONVICTION AND SCEPTICISM

Stamp (undated) recalls teaching a lesson on right-angled triangles. In the first two examples considered - (6,8, 10) and (5, 12, 13) - it was observed by pupils that the area and perimeter had the same numerical value. This led to the conjecture that "this happens every time". Stamp reports that he "denied" that this can be so, and in fact proceeds in the note to deductive demonstration that, with the exception of the given examples, the proposition is universally false.

What do *we* make of this? Tim's reaction to the conjecture was that ifthere were such a connection between the perimeter and area of integer-sided right-angled triangles, then he would already know about it! Therefore (he might reason) there can be no such connection. Whilst this 'mature' mode of reasoning *can* be a reliable guide to induction, it can also be a negative and dangerous reason for scepticism about the remarkable-but-unfamiliar. For example, I [TR] recall vividly that my formal education in mathematics omitted the insight that every quadrilateral tessellates in the plane. The property is so improbable (to me) that I would have expected to know its truth, yet I did not.

Induction is essential for mathematics, but it is not sufficient; the "conquests ofthe intuition" are potentially fallible.

Liz reports two incidents with lower sixth A level mathematics students.

> During a reporting-back session following an exploration of the absolute value function, Lome makes the assertion that the graph of y = If(x)1 is the same as the graph of y = *f(lxl)* for every function **f** I am unsure whether he is right and I try to think of counter-examples. I suggest that he plots y = 12x + 11 and y = 21xl + 1.

Lome had considered five or six examples of functions in coming to this inductive conclusion. I was doubtful, but not because the number of examples considered was too small. Like Tim, I felt that such a striking result would already have been known to me! I also have an image of the graphs of modulus functions that involves points with undefined gradient, where the graph is 'reflected back on itself. My initial, vague feeling of unease formed itself into a counter-example. Then I could see that Lome's examples may have been sufficient in number but of 'the wrong kind'.

Whatever my intuition was which made me doubt the truth of Lome's statement, there was no such doubt in Lome's mind. I can account for this difference in two ways. First, he had less experience of the modulus function on which to draw. Secondly, he was less cautious of inductive reasoning. His schooling had often put him in the position of needing to trust conclusions from inductive reasoning in mathematics without considering the strength of other reasons for conviction.

A few days later ľ recorded what I saw as a similar incident from a class lesson:

> I am talking to the whole class about the way in which they derived the equation of a circle with radius 2 ~ centre (3,5). I have written the equations ~(x - 3i + (} _5)2 = 2 and (x - 3)2 + (y - 5)2 = 4 on the board. I ask 'where did the 3 the 4 and the 5 come from in this (the second) equation?' Trevor replies that the 4 is the diameter of the circle.

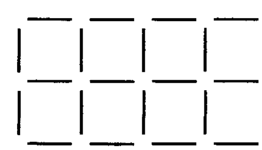
ľ had intended to draw the students' attention to the *structure* of the derivation of this particular equation, with the eventual aim that they would appreciate the form *(x - a)2* + *(y - b)2* = r^2 for the equation of a circle. I was expecting them to base an answer to my question on their recall of the *procedure* by which they had derived the equation. But Trevor seems to be making an *empirical* generalisation from one case, rather than recalling the derivation procedure as I had hoped. I see his statement as a generalisation because he says that 4 is 'the diameter of the circle' and not simply twice 2. His answer relies on seeing that 4 is twice the radius, rather than seeing that 4 is the radius squared, and that it results from the squaring 0Eration which was part of the process of obtaining the equation. He focused his attention on *num ical patterns* rather than *structural relationships.* By contrast, I knew that the constant term in this e uation could not, in general, be equal to the diameter of the circle, since I knew it to be eq al to the square of the radius.

EMPIRICAL AND STRUCTURAL GENERALISA TION

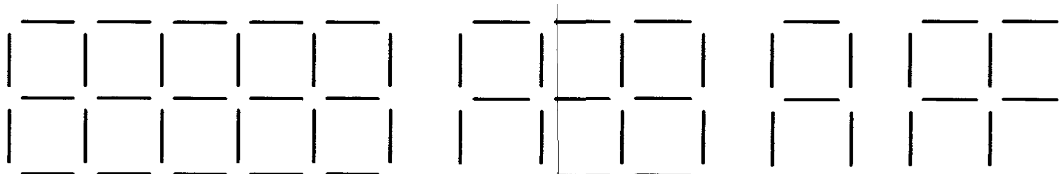
In both of the cases above, the student makes a (sometimes tentative) inductive generalisation. In each case, Liz was sceptical about their conclu ions on the grounds of other sources of conviction. In the second, Trevor generalised from the part· cular circle equation in a way that was different from that intended. We use the terms 'empiric l' and 'structural' to describe, respectively, the form of generalisation Lome and Trevor made and t at which Liz had expected. In using these terms, we emphasise that one form of generalisation is ac ieved by considering the form of results, whist the other is made by looking at the underlying mea ings, structures or procedures.

This distinction is illuminated by Liz's notes o~ the following problem:

> The picture (right) shows a rectangle made up of two rows of four columns of squares outlined by matches. How many matches would be needed to make a rectangle with *R* rows and *C* columns?



When I first worked on this problem, I decided ~o simplifY by holding the number of rows constant. I held *R* as 2 and produced a series of diagrams!such as these:



From my diagrams I produced this table of results:

No. of columns (C)	No. of matches (M)
4	22
5	27
3	17
1	7
2	12

I saw that the results in my table fitted the rule $M = 5C + 2$. My trust in this formula for all positive whole number values of C was based first on the results in my table. Secondly, I was confident in it because it was of the form I was expecting. By this I mean that I expected a relationship to exist between M and C , and previous experience led me to expect the relationship to be linear.

Next I changed the value of R to 3 and, with the aid of one or two diagrams, convinced myselfthat M and C now fitted the rule $M = 7C + 3$. Similarly, I found that, for $R = 4$, $M = 9C + 4$ and, for $R = 5$, $M = llC + 5$. For these, I needed fewer diagrams and tabulated results, because my conviction about these formulae from sources other than my table of results was greater each time. Having established that linear relationships held for $R = 2$ and $R = 3$, I needed only two results in the case $R = 4$ in order to be convinced that I had the correct formula.

Now a pattern was emerging that suggested that a general rule was $M = (2R + I)C + R$.

Again, in moving from these separate formulae for different values of R to one which incorporated variations in R, I based my conviction first on the four formulae I had identified in the special cases $R =$ 2,3,4,5. But I had also the anticipation that such a general formula would exist, would be linear in R and in C and would be symmetrical with respect to the two variables.

Finally, seeing beyond the particular numerical features of a diagram so as to perceive it as a 'generic' representative ofthe general (see below), I was able to see that I could count the number of vertical and horizontal matches as follows:

> there are $C + 1$ columns of vertical matches, each containing R matches; there
> are $R + 1$ rows of horizontal matches, each containing C matches; therefore
> there are altogether $(C + I)R + (R + I)C$ matches.

This line of argument confirmed the rule which I had arrived at empirically. The argument is *structural,* because it is based on a way of counting the matches in this configuration. By contrast, my first line of argument is *empirical* because it is based (predominantly) on a pattern in the table of results: it argues 'for small values of Rand C the number of matches is given by $(C + I)R + (R + I)C$ so it seems reasonable that this will be the case for all positive whole number values'. Inductive reasoning of this first kind can be a useful way of *producing* conjectures, but in the absence of other sources of conviction it may point to erroneous conclusions.

GENERIC EXAMPLES

Sometimes, structural generalisation can be achieved by a type of proof by generic example. This mode is discussed in the literature (Mason and Pimijl, 1984; Balacheff, 1988), but little attention has been given to the pedagogy or the epistemic effectiveness of such an approach.

> The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number. (Mason and Pimm, 1984, p. 284)

For example, a generic proof that $\sum_{n=1}^{2k} n = k(2k+1)$ - for all positive integers k - might be effected by reference to the case k=10, showing how the sum can be evaluated as ten pairs (1 and 20, 2 and 19 and so on), each with sum 21. Despite the generic pedagogic *intention* behind such an argument, it may not necessarily be received by the student with the intended generality. Trevor's perception of the equation of the circle with radius 2 and centre (3,5) is a case in point.

Our evidence in this respect is mixed: some undergraduate students in the first term of a Mathematics/Education course were introduced to the well-known 'Stairs' investigation. This problem concerns the number of ways of ascending a flight of *n* stairs in combinations of ones and twos. The Fibonacci sequence readily emerges in the data, and these students were asked to consider why this is the case - in effect, whether the obvious inductive inference is as valid as it is convincing. One student, Kim, gave an account of why it is that the number of ways for 6 stairs will be equal to the sum of the number of ways for the previous two numbers of stairs. To investigate whether Kim's explanation was perceived as particular or generic, the students were asked to complete the questionnaire below (with spaces for students to write their responses) ..

CLIMBING STAIRS IN ONES AND TWOS

Observation: The number of ways for 6 stairs [**13**] is equal to the sum of the number of ways for 5 stairs [**8**] and the number of ways for 4 stairs [**5**].

Explanation: This is because, in ascending 6 stairs, the first step must be a one or a two. If it is a one, there are 5 stairs left, and there are **8** ways of climbing these 5 stairs. If it is a two, there are 4 stairs left, and there are **5** ways of climbing these 4 stairs. Therefore there are **8+5** ways of ascending 6 stairs.

1. Are you happy with the above explanation i.e. is it convincing?

2. Does the above explanation help to convince you that the number of ways for 15 stairs will be equal to [the number of ways for 14 stairs] + [the number of ways for 13 stairs]?

   If you answered YES, why does the explanation for 6 stairs convince you for 15 stairs?

   If you answered NO, what would you need, in order to be convinced?

3. Does the first explanation [for 6 stairs] convince you that the number of ways *for any number* of stairs will be equal to the sum of the number of ways for the previous two numbers of stairs?

   If you answered YES, why does the explanation for 6 stairs convince you for any number of stairs? If you answered NO, what would you need, in order to be convinced?

Of 17 students, the (anonymous) responses of 15 indicated that the particular example - the explanation for 6 stairs - was, for them, generic in relation to other numbers of stairs. The following responses are typical:

[Student A] If you start with a one, you have 14 left. If you start with a two, you have 13 left. So the sum of these two will form the same formula as for 6 stairs.

[Student B] We could re-write the explanation in terms of *n, n-1* and *n-2* where *n=6* stairs in the explanation and *5=n-1* and *4=n-2.* So we see we had the correct method in the explanation above.

Of the two who were unconvinced (questions 2 and 3), one was explicit about lack of conviction about the initial particular explanation (for 6 stairs), being slightly unsure that it accounted for all possibilities. The other seemed to require further confirming instances of the generalisation before s/he could "accept it".

SUMMARY

We recognise that inductive reasoning of a quasi-empirical character motivates students, and lends an authentic air of discovery to the mathematics classroom. Such activity may transcend empirical speculation if explanation is available to the student as a structural generalisation of some kind. A generic example which successfully "speaks the generality" (Mason and Pimm, 1984, p. 284) for the audience has the quality of such a structural generalisation.

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MATHEMATICS, LANGUAGE AND DERRIDA

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Abstract: *Derrida' s revolutionary work in the study of language has seriously challenged the way in which we see words being attached to meanings. This paper makes tentative steps towards examining how his work might assist us in understanding the way in which our attempts to describe or capture our mathematical experiences modify the experience itself. In doing this we draw on the work of Derrida and John Mason in locating possible frameworks through which to conceptualise the relationship between language and mathematical cognition. It concludes that mathematical meaning never stabilises since it is caught between the individual's on-going experience and society's on going generation of societal norms as manifest in its use of language, in particular, those penaining to society's view of mathematics. That is, mathematics, language and the human performing them are always evolving in relation to each other.*

Recently, there was a conference dedicated to the work of Jacques Derrida Derrida was in attendance. A colleague and conference attendee, Antony Easthope, described how Derrida himself patiently sat through numerous papers speaking of his work without passing any comment. However, at the end of the conference Derrida made his own presentation. Having declared his delight to be in a conference celebrating his work he was, nevertheless, uncomfortable listening to so many people describing his work. He spoke of how, in attending the conference, he had experienced a sensation of being already dead. Having witnessed numerous attempts to sum up his work and integrate it elsewhere had made him feel as though his work had already been frozen for eternity, as if people were no longer seeking his present thinking.

In mathematics, as we survey the offerings of our students, maybe we are susceptible to the same sort of premature encapsulation, wrapping things up as we get some semblance of the thing we seek in their work, torn as we are between encouraging the gradual development of their own individual mathematical understanding, whilst ensuring they meet the social requirements of knowing certain specific ideas. Such difficulties can also arise as we attempt to capture the flow of our own mathematical thinking in to some sort of fixed form for the purposes of sharing. We can never fully express what we see. How indeed can mathematical thinking, a highly temporal commodity, be organised with reference to more stable entities, mental or otherwise? Can we freeze ideas, hold them still while we look at them? Addressing these issues is far from easy since the whole notion of finding stability is complex. In attempting to frame mathematical ideas what sort of loss do we experience?

This paper considers the relationship between mathematics and language, making particular reference to the work on language by Derrida. In taking this perspective I examine how language functions in organising mental activity and suggest that since language is so fundamental to the social formation and individual construction of mathematical ideas, it conditions all mathematical experience (cf. Brown, 1994 d, in press). In this spirit I argue that linguistic reduction is an inevitable aspect of any mathematical construction, both locating and conditioning broader cogitations. As such, loss is a necessary result of the process of stressing and ignoring that underpins any conceptualisation.

Many contemporary writers on language, like Derrida, see the generation of language as instrumental in the self-formation of society and of the individuals within it. In this perspective, language can no longer be a seen as providing an unproblematic labelling of the world. Analytic philosophy's notion of language picturing reality (e.g. Russell, 1914; early Wittgenstein, e.g. 1961) no longer holds up as an adequate metaphor for the way in which language functions, although such a belief may well still govern the everyday actions of many people. How then can we see contemporary work in language as assisting us in discussing mathematical ideas? Like language, mathematics can be regarded as a social produced phenomenon and, in particular, mathematical activity can be seen as being a form of linguistic performance. For Derrida social phenomena can always be read as a text. Consequently, in this perspective, mathematical activity finds itself subject to the scrutiny of modem day critiques of language which emphasise its situation in history, in culture and in personal accounts. The human subject engaged in mathematics is positioned in a number of co-existing social agendas which flavour the style of engagement. Insofar as we see mathematical meaning being generated in the mind we cannot escape the formative influences on the mind. Also, we cannot partition off a section of the mind and label it "mathematics".

Until recently, the dominant traditions of mathematics teaching have focused on how mathematics is, rather than on how it is seen. Teaching media are customarily treated as if they give access to something actually there. The parameters of mathematical activity are clearly delineated, where the symbols assume an unproblematic relation with the concepts they represent. Recent work in mathematics education research, however, has focused more on how participants experience the mathematics classroom. Such "insider" views of mathematics consciously build in some sort of selfreflective dimension. Since participants are necessarily governed by certain social practices I suggest such views are always embedded in a culture. That is, mathematics only manifests itself in activity governed by culturally specific norms (e.g National Curriculum mathematics or university mathematics) (see Brown 1996 b. It seems insider perspectives are becoming more prominent as the absoluteness of mathematics itself is brought into question. Mason (e.g. 1994) has spoken of researching problems, both mathematical and professional, from the *inside.* Constructivists have focused more on the individualleamer' s understanding the mathematical tasks they face. Meanwhile Skovsmose (1994) has commenced the groundwork in formulating a philosophy of critical mathematics education to examine ways in which discourses operate within mathematics education. Part of his task has been to uncover the way in which mathematics education conceals its intentions beneath the language it employs in declaring its project.

Here I shall focus on the position taken by Mason in associating mathematics with language, as an example of a writer in mathematics education moving away from assumptions of language picturing reality. In fact he sees a clear distinction between mathematical experience and the linguistic description of it. The following quotes give a flavour of his view:

Words generate more words in explanation, but often draw us away from the experiences from which they stem. Mason (1994, p.176)

Express to yourself in action (by doing it) and in words (by talking to yourself or a colleague) a role for continuing the following array ..... Honsberger quoted by Mason (1989, p.3)

On the one hand we have the experience, on the other, the description of it in words. In my conversations with him, John Mason defends the content of his mind as not being reducible to description in words. Whilst I may report on my experience as a mathematician, in so doing, I insert a gap between experience and report, resulting in the precise nature of my experience being rather elusive, being partly lost, at least as regards its capturing in language.

How then can we locate mathematical meanings in relation to mathematical and linguistic performance by humans? Traditionally, the task of the teacher and learner may be seen as sharing a preexisting mathematics not susceptible to individual interpretation. Such an account is governed by notions where the teacher seeks to direct the student's attention to a specific way of seeing an objectively understood mathematics. Truth is embedded within the mathematics and the student seeks to locate this. Meanwhile, "insider" views of mathematical problems, as exemplified in the work of Mason, focus on directing the student on a journey around problems and reflecting on the experience of doing mathematics. Whilst the emphasis is more on the student reconciling his or her experience with ways of describing it, Mason argues, the attempt to describe in words might draw the student away from the mathematical experience itself. The teacher's intention is rather less didactic but this may not necessarily imply a less conventional view of the underlying mathematics. Mason follows Gattegno in seeing truth gravitating around personal awarenesses, that is, truth is located in the mind of the
indi vidual.

So where mathematics is located? In more traditional views the mathematical meaning is independent of individual human performance. Meanwhile, according to Mason, the emphasis is on the individual human's personal awarenesses of mathematics. Nevertheless, both appear to see the description of mathematical activity in words as being outside the realm of mathematics itself.

Contemporary accounts of language provide a rather more transcendental view of language that infiltrates, whilst coalescing, the reality it serves. I wish to argue that the framing of mathematical experience in words by individuals should be seen as an integral part of the mathematics itself, inseparable from less visible cognitive activity. I shall briefly mention the hermeneutics of Ricoeur and Gadamer before developing Derrida's more radical post-structuralist position.

Ricoeur and Gadamer assert that experience itself is conditioned by any attempt at a linguistic framing. For them language mediates truth. That is, whilst they are happy with the notion of truth, they see this truth as being obscured by our attempts to access it. The chief consequence of this for our current analysis is that mathematical experience and description of it in words are drawn closer together. Phenomenology has offered an approach, supportive of their hermeneutic framework, which assists us in discussing how the individual confronts and works with mathematical ideas. Here, the material existence of the world is fully accepted but it only presents itself according to some particular phenomenology subsequent to being carved up in a time dependent categorisation by an individual. The material world lights up as it is touched by the human's gaze. Objectivity itself is historically created defined in terms of the way in which the individual consciousness perceives the material. This

partitioning of the material world into phenomena is closely related to the descriptions made in respect of it Mathematical objects then, present within such thinking, are not unproblematic entities for all to see, but rather, are understocx.l differently by each individual. The distinction between such phenomena and the perception of them is softened with phenomena and perception evolving together through time. In this perspective mathematical ideas, as located through notation, are not endowed with a universal meaning, but rather, derive their meaning through the way in which an individual attends to them.
Thus mathematical "object" and human "subject" are seen in a more complementary relation as part of each other. The emphasis in this phenomenological formulation is on the individual's experience of grappling with social notation within his or her physical and social situation. This provides a framework, seen from the individual's point of view, in which the distinction between the individual and the social is softened. In building his or her understanding, the individual is obliged to work through the social filter of language. My strategies for making sense of and acting in the world are always underpinned by cultural stylising derived through language, whether I be mountain climbing, dancing or doing mathematics. All such activities can be seen as specific discursive "spaces" (Stronach and Maclure, 1996, p. 262). For a fuller discussion see Brown (1994 a, 1994 c, 1996 a).

Derrida (1978, pp. 278-293) similarly claims that you can only observe the linguistic performance of others from the home base of your own linguistic frame. One is always positioned within culturally derived ways of seeing and so experience itself is textual insofar as it is understocx.l through inherited schemata embedded in language usage (cf. Brown, 1994 b d). In short, language is always about a world already conditioned by language. Any human performance can be read as a "text" in the philosophical sense of the word. Indeed, Derrida famously asserts (1976, p. 158), "there is nothing outside of the text", nor are there truths to provide points of anchorage. He sees differential structures as being inherent in explicit language, consciousness and unconsciousness. Like Lacan (1977) he identifies this as a general feature of the mental world, with both conscious and unconscious being structured like a language. The mental world, so seen, is a system of differences, part of which is claimed by explicit linguistic structuring (Derrida, 1982). It should be stressed, however, Derrida does not dismiss the experience itself, rather, experiences are in a constant state of flux conditioned by attempts to associate them with a never ending linguistic flow. He would see mathematical involvement as necessarily textual, brought about through human partitionings of the world - a framing that is, in a sense, already there, brought about through cultural linguistic heritage. "My own words take me by surprise and teach me what I think" (Merleau-Ponty, quoted by Derrida, 1978, p. 11). Derrida builds on this quote in discussing how inscription in words (and maybe also in symbols, in diagrams) orients psychologically produced phenomena. If I may risk using his own, rather slippery, words:

> If writing is inaugural it is not so much because it creates, but because of a certain absolute freedom of speech, because of the freedom to bring about the already there as a sign of the freedom to augur. A freedom of response which acknowledges as its horizon the world as history and the speech which can only say: Being has already begun .. (Writing) creates meaning by enregistering it, by entrusting it to an engraving, a groove, a relief, to a surface whose essential characteristic is to be infinitely transmissible. Not that this characteristic is always desired, nor has it been; and writing as the origin of pure historicity, pure traditionality, is only the tel os for a history of writing whose philosophy is always to come. Derrida (1978, p. 12)

I take Derrida to mean, crudely, that inscription in writing functions closely in relation to the psychological phenomena it locates and, indeed, becomes part of it. In reading Derrida one never gets to what he means but rather one experiences the on-going sensation of being moved on before you are ready. His words never frame the final version of his "present" thinking. In this respect, Derrida's position is not that far away from the the more moderate line of Gadamer and Ricoeur who peffilit an on-going renewal within the co-evolution of phenomena and perception. However, Derrida's refusal to allow any anchorage in truth makes his work quite distinctive and more radical in its ability to reject orientation around universal structures.
Derrida's position takes language well beyond its traditional scope towards embracing the whole of human experience. Objections sometimes arise when we attempt to nudge language into this extended domain. For example, on the surface at least, such views appear unsatisfactory to those who wish to defend the power of their own mathematical experiences as being outside the realm of language. Nevertheless, Mason does speak of manifestations in the "outer" which have some sort of association with "inner" experience, indeed he seems distinctly Buddhist when he suggests that we need to acknowledge "a world of experience that is not material, not phenomenal, but inner, with access through what we are able to read in the outer" (Mason, 1994, extended version, p. 7). This, I feel, invites a degree of compatibility between his understanding and the line taken by Derrida.

We need, however, to ask about the nature of this association and question how these outer manifestations attach themselves. Are they like the tips of icebergs (i.e. part of the thing being signified) or like road signs (i.e. separate to the thing being signified)? I suggest words, diagrams and other manifestations of mathematical activity, can be seen as functioning in either way, according to current interest and the emphasis one assumes. Indeed they may be seen as two points on the hermeneutic cycle connecting ways of seeing experiencing and ways of describing (Brown, 1991). The physical environment, for example, is textual, in Derrida's sense, insofar as the human eye organises it differentially. Thus "seeing" is always in relation to an a priori conditioning. Any attempt at inscription reflects this broader but maybe more elusive differentiality. As with Saussure (1966), Derrida sees the signifier/signified duality as inseparable. But as with Lacan (1968), Derrida sees relatively stable signifiers being associated with a fluid underbelly, comprising a signified field which sweeps out to occupy the whole of consciousness, and indeed, the unconscious. Both presence and absence are located by the signifier. The loss incurred in the attempt to articulate remains attached to the signifier seeking to replace it. Meanings are derived only through retrospective examination of the flow of signs. The component signifiers do not have implicit meanings, only relational associations with other signifiers in the chain. There are no independently existing meanings in the chain since any attempt to frame in words, any attempt to "mean", creates a gap between "being" and attempts to explain it. Lacan speaks of an indefinite sliding of meaning to convey the "impossibilities" of attaching one word with one meaning. We have no truths to provide orientation apart from those generated through this system of differences. Derrida (1981, translator's introduction, p. ix) suggests that selfpresent meanings are illusions brought about through repressing the differential structures from which they spring. However, as a note of caution Derrida seems to have back-tracked a little from the extreme way of thinking many associate with him:

... it was never our wish to extend the re-assuring notion of text to a whole extra-textual realm and to transform the world into a library by doing away with all boundaries, all frameworks, all sharp edges (Derrida, 1991, p. 257).

In describing mathematical experience we may suspend the "presence" of the experience, in a sense, but the experience itself was textual (i.e. understood differentially) and thus already a suspension, so no more nor less the "real" experience. There is no experience outside the text, only a retroactive constructions of it asserted by the individual. To make strict distinction between experience and description of it in words, as Mason appears to, requires a relatively restrictive view of language. A more phenomenological account of language sees the spoken word as rather more like a tip of an iceberg, that is, as part of the thing it signifies, in particular, the silent cognitive activity taking place around it. Whilst Mason's distinction might offer a valuable rhetorical device in initiating or analysing mathematical performance, such a distinction suppresses the historicity endemic in anything commonly recognised as mathematical performance, or even mathematics itself, and thus obscures the values associated with this (cf. Derrida, 1989). In particular, the linguistic forces driving (and being driven by) mathematical constructing get squeezed out of the picture. Mathematical constructing, I would suggest, is always linguistic to a degree, oscillating in a hermeneutic circle, between more or less sturdy linguistic frames.

Both hermeneutics and its radical form, post-structuralism offer accounts which, through being more flexible in their understandings of language, engage with the material qualities of the world. Phenomenology, as present in the hermeneutics of Ricoeur, accepts the material world but intercepts perception before it assumes shared notions of categorising this material world into objects. Poststructuralism meanwhile uses language itself (or more accurately textuality, i.e differentiality) as its home base and so subjects meet in their shared use of the manifestation of this in speech or writing. In both these theoretical perspectives, associations between language and reality resist stability between signifier and signified. Rather, both reality and language are caught in an historical process of mutual formation which is never complete, nor even pauses long enough for one to map the other. In such a perspective, the historicity present in both the genesis and the current performance of mathematics is recognised. As such there is no mathematics outside language.

In this perspective, the learning of mathematics moves away from being concerned with recreating existing ideas but instead emphasises the tightly knit relation between language and understanding. On this point Mason and Derrida seem close. In assuming the teacher's task himself, Mason is concerned with enabling his students to generate their own mathematical experiences. That is, he does not explain the mathematics in his head but rather, initiates an activity which he hopes will enable his students to experience some mathematics and in this, perhaps, encounter certain ideas. He sees learning as a journey of self discovery.

" . .it is important to re-search, re-collect, re-connect, re-learn, re-integrate, and re-cast insights in the discourse of the times. I see working on education not in terms of an edifice of knowledge, adding new theorems to old, but rather as a journey of discovery and development in which what others have learned has to be re-leamed, re-integrated and reexpressed in each generation" (Mason, 1994, p. 177).

Ideas are not inherited prepackaged and intact, but rather, each new generation will engage in tasks that give rise to new understandings of what might be seen as old ideas. There is a need to work on ideas, they cannot just be "received". This way of thinking bears a striking similarity with some of Derrida's recent work:

> Inheritance is never a *given*, it is always a task. .. there is no backward looking fervour in this reminder, no traditionalist flavour, Reaction, reactionary or reactive are but interpretations of the structure of inheritance. That we *are* heirs does not mean that we *have* or that we *receive* this or that, some inheritance that enriches us one day with this or that, but that the *being* of what we are *is* first of all inheritance, whether we like it or know it or not. Derrida (1994, p. 54).

Whilst the student's task may well oscillate between fitting language to mathematical experience and bringing meaning to language through reflection on one's own experience, both experience and linguistic production forever continue, resisting attempts to settle on a particular version. Mathematical meaning never stabilises since it is caught between the individual's on-going experience and society's on going generation of societal norms as manifest in its use of language, in particular, those pertaining to society's view of mathematics. Mathematics, language and the human performing them are always evolving in relation to each other. There is no final *version* to be learnt, since we lack universal truths to hold this *in* place. I conclude with a quote another post-structuralist thinker, writing shortly after the student uprising *in* Paris *in 1968:*

> Just as psychoanalysis, with the work of Lacan, is in the process of extending the Freudian topic into a topology of the subject (the unconscious is never there in *its* place), so like wise we need to substitute for the magisterial space of the past- which was fundamentally a religious space (the word delivered by the master from the pulpit above with the audience below, the flock, the sheep, the herd) - a less upright, less Euclidean space where no one, neither teacher nor students, would ever be in *hisfinal place* (Barthes, 1977. p. 205).

> In short, within the very limits of the teaching space as given, the need is to work patiently tracing out a pure form, that of *afloating;* a floating which would not destroy anything but would be content simply to disorientate the Law. The necessities of promotion, professional obligations .. , imperatives of knowledge, prestige of method, ideological criticism - everything is there, but floating (Ibid. p. 215).

NOTE

I am grateful to John Mason, Ian Stronach and, in particular, Dave Wilson for extensive comments on an earlier draft.

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Flipping the Coin: Models for Social Justice in the Mathematics Classroom.

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Abstract: *This paper offers a definition of "social justice", a term which is in great danger ofbeing over used and thus losing any meaning, as well as exploring the models* 1 *am working with the theorise the issue of social justice in and through mathematics classrooms and mathematics teaching. The paper draws heavily on the work of Rawls* (1971) *as well as recent workfrom McCarthy (1990), Brandt (1986), and Marion-Young (1990).*

For me the term social justice became important as I realised that activity I was involved in under the guise of anti-racist work drew heavily the beliefs I held about education in general and about mathematics education in particular, which may be subsumed under a heading of democratising schools and classrooms. Here the term social justice may define a way of working which accounts for and works the links between oppressions, inequalities and exploitation's which we see inside and outside our schools and classrooms.

However, recently, Social Justice has become capitalised. Social justice is defined by the 'Report from the Commission on Social Justice' as a hierarchy of four ideas:

- *The equal worth of all citizens.*
- *Each citizens equal right to meet their basic needs of food, shelter and other necessities.*

•*The need to spread opportunities and l?fe chances as widely as possible.* •
The need to eliminate where possible unjustified inequalities. (Commission for Social Justice, 1994. :pl)
the use of language such as 'where possible' seem to offer little hope of any real commitment to structural change, and the concept of 'unjustified inequalities' reflects the Victorian view of the 'feckless poor'.

For me social justice represents a shift in thinking away from equality in classrooms. Equality suggests a norm that we work towards. It does not easily accept and value difference although attempts were made through slogans such as 'equal but different' to address this issue. Maybe social justice can be seen as the beginning of a theory around this slogan.

Social Justice is to also do with power. It is do with how individuals and groups of individuals can feel powerful or be made to feel powerless. It is to do with individual,s feeling in control of decisions which affect the way they live their lives. We feel as though an injustice has been done when someone takes a decision which affects us personally or emotionally and with which we disagree but against which we have no power to argue. Social Justice in this sense is linked to the power to make life choices without being denied access to particular life chances through discriminatory practices, Social Justice is also having the power as well as the right to fight practices we perceive as unjust.

Where does mathematics fit into all this. As a child of 11 I was taught in the 'top' group in my Primary school. On Fridays we had a mental arithmetic test. The child who scored highest in the test sat in desk 1 (next to the teacher) for the following week, the child 'coming second' sat in the next desk and so on. After a week or two I realised that by getting 2 or 3 questions wrong I would be about fifth in the test and get to sit by the door. I could engineer this result as I was confident that I knew all the correct answers and so could deliberately make my 2 or 3 errors. I offer this as a metaphor for confidence and skill in mathematics carrying with it power over life choices. A teacher on a course brought this memory to the front of my mind when she told of a similar experience although her experience had left her feeling powerless and out of control. She felt she could not 'do' mathematics, Fridays brought with them anxiety and panic. She knew she would do badly in the test, would be made to sit in a 'lower' desk and worst of all that her mother would see she had failed when she collected her after school. I also note the gender of the tellers of the two stories.

The issue of 'power' is becoming increasingly important in our research community - I cannot even begin the discuss this here. However the theory of power presented here may be seen as more essentialist than a Foucauldian view as it draws heavily on the work of Rawls - who feels able to define individuals as 'least powerful' in certain institutions. A view which I have sympathy with

Although Rawl's work has been criticised heavily for its romanticism, for its detachment from historical and political realities, and for its neglect of the factors of sex, race and class it offered me a useful starting point for my exploration of social justice and education. It also has allowed me to look at ways in which institutions of education can be examined in order to move towards 'more just' ways of working. It opens up possibilities for change.

Rawls uses the metaphor of 'halving an apple' to explain the basis of his theory. If two people are sharing an apple, one person cuts the apple and the other has the choice of which half they want. The theory being that the first person will be as fair as possible in cutting the apple in order to ensure they receive a fair share. The creation of his Utopia takes place through a discussion by a group of people shrouded by a veil of ignorance - they do not know whether they are male or female, what their ethnic background might be, what their family situation is or what their historical backgrounds are. He suggests, taking perhaps a rather pessimistic view of human nature that this group would seek to protect themselves from harm in the new society they were creating. To build ajust society, we should create a society as if our enemy would choose the position in which we are placed within that society. This argument is extended to say that inequalities in distribution within institutions or societies are only just ifthey benefit the least well off within that institution or society.

Perhaps Rawls offers us ways to critically examine our institutions and our classrooms. Can we apply his tests of justice. Do decisions we take as to the arrangements within our classrooms and our institutions always benefit the 'worst off amongst our learners? Would we feel comfortable if we thought that our enemies could decide where to place us, or our own children in order for us to learn mathematics within our schools. If the answer to either of these questions is no, in what ways would we alter what we teach, or the way that we teach it to accord to Rawlsian justice?

A Rawlsian interpretation of social justice only allows 'unfairness' within an institution if the 'unfairness' benefits the worst off within that institution. Another measure of social justice put forward by Rawl's is the right of all people to decide on a rational life plan which is designed to permit *'the harmonious satisfactions of his (sic) interests'* (Rawls, 1971 : 93). A rational life plan is one which *'cannot be improved upon, there is no other plan which, everything taken into account, would be preferable.'* (Ibid. : 93). Education in general and mathematics education in particular is clearly important in bringing this 'rational life plan' to fruition. Academic mathematics qualifications are used as a filter to future career prospects and any 'rational life plan' involving moving into a profession must include success in mathematics. However there are clearly many injustices within the assessment structure in our schools, even were these to be eradicated we have already seen that equal qualifications do not mean equal access for all to future career choices. Rather than suggest that all must be given an equal opportunity to succeed academically at mathematics within the present structure this suggests to me an arena of research is that of the assessment processes used by mathematics educators in our schools. How can this be transformed to meet the conditions of social justice.

Rawls also suggests that within a socially just society, individuals and groups would feel able to participate actively in the democratic organisations within that society. This view of social justice raise many questions with regards to the mathematical education we offer in our schools. Have we begun to address the idea of'mathemacy' as Ole Skovmose (1994) calls it, which sees the possibility of constructing a learning of mathematics which is designed to support learners in their development as reflective adults capable of using their mathematics to critique and challenge structures within society. If the purpose of teaching mathematics was to enable our learners to construct a better society we would clearly have to re-evaluate our notions of curriculum and pedagogy.

It would seem to me that these ideas are entirely compatible with the desire that mathematics classrooms should be places where we educate both for mathematics and for a society based on ideas of social justice. It also suggests an acknowledgement that mathematics rather than being a tool to be used to interpret and explain the world around us is also used to create our world. I think this is an important point. We cannot move towards a mathematics for justice without questioning our notions of the nature of mathematics and the nature of mathematical knowledge. Indeed by exploring the social perspectives of mathematics education we begin to question many of the unjust practices present in our schools today and the search for alternatives begins.

However although this Rawlsian perspective offers useful models on which to build a theory of social justice and mathematics education it views the values of justice and autonomy as moral issues detached from everyday hwnan behaviour. This is challenged by Carol Gilligan (1988) who asserts that for many women, the notion of care is a key to the way that moral decisions are made. The push for autonomy within a society leads to a detached view of an individual, living within a hierarchically ordered society, whereas the values of care and attachment create a world of individuals within an attached network of relationships. Incorporating the idea of 'care' within a social justice framework offers extra possibilities for transformation rather than adaptation, and again moves from equality as equal turns to social justice as a transforming power.

Gilligan's work has been criticised by Paul Ernest and Patricia Hill Collins (1990) for relying entirely on middle class, white women within its sample. However by attempting to pull together the links between the inequalities, exploitations and injustices suffered by different groups we begin to move towards a coherence which allows us to operate as researchers. So any theory of social justice must include notions of care and connection with familial and cultural roots, if it is to be a useful model. Clearly a mathematics for social justice must include a perspective of care. We must not strive simply to produce autonomous, independent human beings, ready to play an aggressive role in pushing forward the domestic economy, or confident to take their place fighting for a place in a new job market but must also look towards pedagogies in mathematics which encourage values of sharing, co-operation, joint labour and skill sharing. Most importantly we must involve multiple perspectives when viewing actions and interactions in our classrooms, we must acknowledge difference rather than foster homogeneity.

As a researcher interested in issues of justice in mathematics classrooms I required arenas in which I may act. Drawing on Brandt (1986) the areas of syllabus, pedagogy and the social and cultural environment of the school/classroom and assessment became areas of interest. These arenas may be summarised under headings of;

- what mathematics do we teach?
- how do we teach?
- what do we value?
- how do we feel?

Additionally Cameron McCarthy (1990) offers four relationships which could be observed if injustices are present in our classrooms; (i) a competitiveness which leads to individuals or groups becoming isolated from mathematics and from mathematics learning, (ii) the domination of one group over another in the classroom, of one teaching style over another, of teacher time, of resources and so on, (iii) exploitation by dominant groups of weaker groups or of learners by teachers and, (iv) the cultural selection which can take place within mathematics practices.

Finally Iris Marion Young offers five facets of oppression. That is individual experiences which are shared by all individuals belonging too oppressed groups at one time or another. These facets do not exist separately but can be used to observe instances of injustice within mathematics classrooms. She defines these as, powerlessness, violence, exploitation, marginalisation and cultural imperialism.

These three viewpoints can be used to explore how schools in general, and mathematics practices in particular, both in terms of content and teaching styles, at worst create, or more usually, fail to challenge injustices within society. By using these facets of oppression as filters through which to view classrooms I am working towards building a model of a mathematics curriculum for social justice in our schools. A project which is far from complete and often seems almost impossible to carry out. However I share the view of others that such a project is worthwhile and even important. It has offered me a methodology for my research which ties in with those values I held dear as a classroom teacher. I am not attempting to explain why classrooms work in the way they do - that seems like an even more impossible task, nor am I trying to offer a view of learning which can be generalised to improve our teaching. I have a great suspicion of people who tell me how children learn - it is always to easy to find a counter example. At the moment this framework is offering me a way of asking questions which seem fundamental to me as a mathematics teacher and as human being.

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A Co-spective Way of Working

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1. Introduction

For those who have followed our work over the last few years, this paper may come as a surprise. Our previous work has moved in a clear direction towards developing and using our own theory of learning (Duffin and Simpson, 1995). In this paper, however, we have paused from that goal to reflect on *how* it is that we are working and in what sense, if any, what we are doing is valid as research.

Early in our work together, we found ourselves talking abour "our way of working". More recently, we began to notice that we were often replacing that phrase with "our methodology". When we began to notice ourselves using the two phrases interchangably, we were led to one obvious, but important question:

> how is a way of working different from a methodology?

We had also become aware, through responses to our work, that people were interested in what we did and how we did it, and that they were relating what we were saying to their knowledge of related literature. In order to site our work in relation to that of others, we found that we needed both to make explicit what it is that we were doing (and which had initially come about as a natural response to our different viewpoints on a shared interest) and to see how it might fit with our initial views on methodology. So, in this paper, we consider what a methodology might be, discuss our way of working, then try to relate that way of working with the methodological issue that we raise. In addition, we will highlight some of the important questions which were raised by the discussion which followed our presentation of these ideas and which, inevitably, have taken our ideas further forward.

2. Methodolatry and Our First Thoughts on Methodologies

When we first began to consider what we might mean by a methodology, we came across what became an important notion for us: not because it moved our thinking on in the concepts that it provided, but that it stimulated a chain of thought about the research process which became a central focus in our attempt to site our work amongst that of others. This came originally from the work of Daly (1973) which we first met in the context of women's involvement in both learning and the research process discussed by Belenky et al (1986). The notion which we met was embodied in the single word 'methodolatry', which Belenky et al interpret as the idea that there is a standard set of acceptable methodologies and - since the questions that can effectively be asked are, to some extent, determined by the methodology used - those who wish to ask different questions or use different methodologies are "rendered invisible".

While we find the notion of methodolatry in itself a useful one, our train of thought was caught by part of the interpretation: that methodolgies, to some extent, determine the questions which can be asked. Our initial idea was that this was just part of a two-way process: that questions can determine methodologies and that methodologies can determine questions. Indeed, we might argue that such a clear-cut distinction may not reflect what we believe actually occurs: that questions and methodology are so bound up together that there is a constant ebb and flow between them in the research process, as both become refmed over a period of time.

As we continued to clarify our thoughts on this matter, we began to realise that this notion of 'ebb and flow' went beyond the clarification of the question and method. We introduced for ourselves the notions of the domain and range of research; that is the subject of the research (who or what is being researched) and the users of the research (who can or will use the results of the work) and we began to see that there are possible influences which may come from these. For example, if the researcher wishes (as we do) to make their research available for teachers to use in their classrooms, the form that the research results take must be one which can be accessed by teachers. This, then, influences the kind of questions that can be asked and thus the focus of the research itself.

The result of this chain of thought was a list of five questions which we felt could provide us with tools to site our work amongst that of others, if we could answer them. They were:

- who or what is being researched?
- what questions is the research asking?
- what form do the answers take?

•who can or does use the answers? •
what do they use them for?

To indicate the fundamental notion, for us, of the ebb and flow of influence between these questions, which is lost when we write them as a list with an implied direction, we produced figure I:

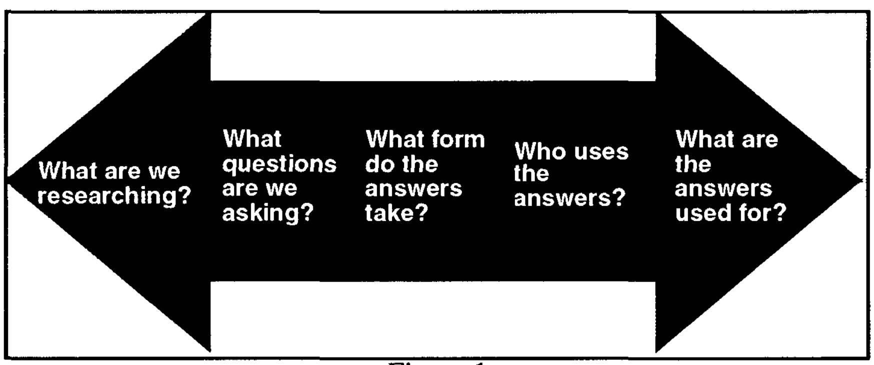

Figure 1

But, before addressing some of these questions from our own viewpoint, we need to make explicit the details of our own way of working.

3. Our Way of Working

When we first began working together the idea that what we were doing might be called 'a way of working' was not in our minds; that we might be doing research was even farther removed from us. Initially we merely focussed, from our own experience as learners, teachers and teacher trainers, on any incidents which caught our interest and through which we hoped to arrive at some mutual understanding. This was purely for ourselves.

Indeed, we began working together when we discovered a shared interest in a single piece of work done by an eight year old girl (Duffin and Simpson, 1991). It was our realisation that we had viewed the piece of work from two totally different perspectives that generated the original stimulus for continuing to discuss such incidents.

One of the features of these early discussions was that we found we were constantly referring back to our own learning as well as calling upon our current perceptions arising from our own work in different spheres. We became aware of the importance of our own learning to what we brought to the discussion. We also became aware of what was important to us in the sharing: the way in which this drew attention to differences of perception and, in the talking, enlarged and changed the perceptions of both of us.

It was only at a later date that our attention was drawn to the contrasts between the approaches of Piaget (working through the individual) and Vygotsky (seeing the genesis of learning as rooted in interaction between individuals) as we listened to colleagues more experienced in research than ourselves. For us, both our individual ways of viewing what we experienced and the important changes and developments arising from our interaction became cornerstones in our own development.

Only very recently, as our thoughts turned to ideas about connecting our work with that of others, that we began to feel the need to try to be explicit about our way of working. We identified three essential characteristics:

- Introspection
- Co-spection
- 'As if from the inside'

By introspection we mean constantly seeking to discern our individual perceptions of experiences, both past and present, and our reactions to them. We suggest that looking at ourselves from the inside gives us an access to the mental processes of a learner that we do not have in studying other people.

We use 'co-spection' to mean the sharing of our own personal reactions to experience with the deliberate intention of using samenesses and differences to further both our individual and our shared perceptions. By 'as if from the inside' we mean that we try to approach the observation of the actions of others, usually in some kind of learning context, from a viewpoint which takes into account the individual learner's likely perception of any experience they may encounter as far as that is possible and as fed by our own introspection and cospection in our way of working.

It is noticeable that much of the language which we use has come from reflecting upon the work of John Mason and adapting our perception of his ideas to our research process. It is Mason (1994) who points out that the sole use of introspection as a research tool in psychology was strongly challenged by, for example, Watson (1913) and the excesses of treating personal, inward looking accounts as unchallengable partially led to the development of 'objective' behaviourism. It is clear in Mason's development of 'intra-', 'inter' and 'extra' spection (from which we developed the word 'co-spection') that he wishes to distance himself from these problems.

However, for us it is the synthesis of all *three* characteristics - examining our individual responses to experiences, sharing them closely with another (who tends to respond differently in many ways) and using the similarities and differences of the internal processes we get from these to consider our observation of others 'as if from inside' which constitutes our way of working. The sharing through 'co-spection' and the examination of any theory we develop through the 'as if from inside' process provides us with a challenge to our individual, internal accounts which introspection alone cannot do.

The phrase 'as if from inside' also comes from a reflection on some of Mason's work. In Mason (1987) he gives a partial classification of researchers:

> Weare all trying to model or describe the inner world of experience. Some of us proceed by contemplating and studying other people, or by studying ourselves as if from outside; others proceed by contemplating and studying ourselves from inside.

It seems clear to us that there is a partial symmetry here, relating the two dimensions of studying ourselves/others and study from the inside/outside. But while Mason relates ourselves to both inside and outside study, he only mentions the notion of studying others from outside. It appears to us that attempts to look at our observations of others by considering what internal structures may be being brought to bear (having looked at our individual internal structures and had the other's shared closely) fits this extra classification of studying others 'as if from inside'.

This element came across clearly to us in an early stage of our work (even though the vocabulary came later). It came about when we were developing our theory of learning, *Natural, Conflicting and Alien* (Duffin and Simpson, 1993). Initially, we only had the two concepts of 'natural' and 'conflicting' but it was in examining an incident about a seven year old boy that we became aware that something which, to us, seemed like a contradiction that should have caused a conflict in the leamer, seemed completely to bypass the boy. This incident led us to consider what, in the mental processes of a leamer, might lead to this kind of response and for which we coined the word Alien to describe that kind of experience. It also led us to realise that an essential feature of our work was that we were trying to observe learning incidents that came our way as if from the viewpoint of the learner - as if from the inside.

4. Connecting Our Way of Working

Our long term aim, then, is to try to connect our own way of working with the five questions (in figure 1). Before addressing this issue, however, it is important to note how our own work has shown the ebb and flow which we think is evident in much research, even though it may be hidden in the papers which many researchers produce.

Looking back at our earliest work together (in trying to explain why our reactions to that single piece of work differed) we can see that we began with a personally motivated question. This obviously influenced the form which any 'answers' we obtained took, since they were essentially just answers for us. The flow was predominantly from questions to the form of answers. However, as other people began to take an interest in our work, we were encouraged to make those answers available in a form accessible to other researchers (and, because of our own interests, accessible to teachers as well). Thus we were led back from the issue of the form of the answers we wanted to obtain, to the kinds of questions we were asking: the flow had reversed.

The notion of this ebb and flow makes the answering of these questions even harder. Whatever order we tackle them in will surely influence the way in which we answer them and is influenced by our current view of our research. Thus it is important to recognise that we have only just begun to address them and whatever answers we give are extremely tentative.

Perhaps we can start from the question 'who can or does use our research'. Our first attempt at such an answer might be:

> For a considerable amount of the time that we have been working, we have wanted to make our research available to teachers. Part of the reason for this comes from our own feeling that some research which might have an influence over us, both as researchers and as teachers, is inaccessible to us: it is written in a language that makes it clear that its audience is (perhaps exclusively) other researchers in that field. (Again, we may see this as the influence of the question 'what are the answers used for' over 'who uses the answers' which also influences the other questions we are putting forward). So we see our aim as the production of research for three main groups: ourselves (to take us forward in our own understanding of the teaching and learning process), other researchers (to enable them to see our view of learning and compare it to their own and other people's) and teachers (who may wish to use our theory to enable them to model the learning processes of their pupils differently, or who may wish to use our way of working to develop their own models)

In writing that answer it is noticeable that we have had to bring in answers to 'what is the research used for' but also 'what form do the answers take'. Indeed, in coming to just this one partial answer, we have become aware of how threatening these questions can be when they are separated: it appears that the research process entangles them so much that they cannot be dealt with easily on their own. There remains much work for us to do in order to address these issues.

5. Ideas from Discussion

As usual, at BSRLM sessions, the discussion with colleagues who attended our presentation took us forward in our thinking. Their perceptions of their own research in the light of the questions we raised and of the questions in the light of their own thinking about methodology gave a new view on what we are trying to do.

There were clear differences in some of the perspectives of contributors: some were research students who felt that the conventions of writing a PhD thesis gave a tightly structured view of what could be classified as valid research. While they seemed to accept that all research had a substantial element of ebb and flow of influences between the five questions we posited, they saw the thesis as only permitting a linear story (perhaps in just one direction along the diagram in figure 1). Others appeared to see their work less constrained and spoke of being able to open their ideas up to alternative ways of working while conceding that, as students, they may also have had to conform.

In addition, a view came out that seemed to challenge even this. One statement made at the session brought in the notion that "research is systematic enquiry made public" (Stenhouse, 1984). This suggests that the essential feature of work which is deemed to be research does not arise because of its conformity to a *laid-down* rigour, but that validity comes from being explicit about the procedures used; that rig our might come from measuring reality against those explicitly expressed procedures within the work.

Perhaps this gives us the first, approximate answer to our question "how is a way of working different from a methodology?" A way of working becomes a methodology when it is made rigour (through being made explicit) and can justify the intricate relationships between the questions that it asks and the methods that it employs.

Perhaps that is just what we are beginning to do as researchers.

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AN ANALYSIS OF STUDENTS TALKING ABOUT oRE-LEARNING' ALGEBRA: FROM INDIVIDUAL COGNITION TO SOCIAL PRACTICE

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Abstract

In this paper we report on a study with the aim of investigating how afocus on langlloge and meaning can assist students in reconstructing algebraic knowledge. The project is set in the context (?f ongoing work with students in Higher Education who need to develop their understanding oj" algebra ~f they are to make sllbstantial progress within their linde/graduate sllldin. Jhe prr?ject is based upon a belief that students' dWiculties with algebra are lallgllage-reloted We 1101'(' collected extensive data by means (?l videotaped sessions involving the stlldents talking oh()/ft their OWl! understandings (?l algebra. The stlldents involved were drawn/i'om courses ill illl/wl t('ocher education and engineering. This paper presents a detailed allalysis (?l the re·V)()I/ses (if olle stlldent and discusses the ways in which this sh~lted our attention as reseorchers./i·om lookillg ot ollr data from the perspective (?l individllol cO}..,Tnition toward~' one iJ?lormed by social prucflce themy.

Introduction

The Re-Learning Algebra project grew out of the difficulties many students have \Aith algebra which have been observed in the course of working in the arena of Academic Maths Support at Sheffield Hallam University. These students have considerable prior experience with algebra and many have undergone years of drill and practice. They have encountered algebra as both an ahstract topic in its own right and also within various contexts. Therefore any additional help offered 10 sllch students clearly needed to take account of previous experience but also needed to have a different emphasis. An approach which was seen to be successful in practice involved encouraging interaction using group activities in which the students could share their understanding and experiellL:e The activities also addressed the use and development of algebraic language and have previously been reported on in Elliott and Johnson (1995).

Related Literature

The literature cited in this section of the paper helped to formulate our thinking and has informed our discussions during the course of this study. There has been much research on algebra at the school level e.g Lesley Booth (1984), including the studies concerned with school children's errors in algebra and an analysis of these errors leading to a categorisation of types of meaning associated with algebraic notation. Anna Sfard and Liora Linchevski (1994) developed their *tl1em\' oj re[fication* according to which there is an inherent process-object duality in the majority of mathematical concepts. They propose that the development of algebraic thinking is accomplished by means of a sequence of ever more advanced transitions from the operational to the structural. In particular they consider two especially crucial transitions: that from the purely operational algebra to the structural algebra 'of a fixed value' i.e. an unknown and then from there to the functional algebra of a variable. Carolyn Kieran (1989) emphasises the recognition and use of structu re as a major area of difficulty in algebra. She reports that for students, who tend to view the right hand side as the answer, 'the equation is simply not seen as a balance between right and left sides nor as a structure that is operated on symmetrically'. Carolyn Kieran and Nicholas Herscovics (1994) propose the existence of a *cognitive gap* between arithmetic and algebra which can be characterisee! as 'the student's inability to operate spontaneously with or on the unknown'.

We found a different emphasis in the work of Abraham Arcavi (1994) which seemed closer to our initial starting point for this study. He outlines what he terms as 'symbol sense' and which he compares to the notion of 'number sense' which has received far more attention His work is about describing and discussing behaviours and not about defining and describing research on students' cognition and ways oflearning. There is an emphasis on sense-making in mathematics and on recognising meaning.

Theoretical framework

Given the initial aim of this study, which was to investigate how a focus on langua.!-(e ane! meaning can assist students in reconstructing algebraic knowledge, we have sought to develop (I theoretical framework which takes account of this emphasis on language and meaning.

A key influence has been the work of Lev Vygotsky (1962), underpinning which is a Ct~ntral assumption that socio-cultural factors are essential in the development of mind. Intellectual development is seen in terms of meaning making, memory, attention, thinking, perception and consciousness which evolves from the interpersonal to the intrapersonal

In exploring the notions of sense and meaning further some useful ideas were drawn tl'om the field of activity theory. In particular Erik Schultz (1994) offers some interpretations of sense (111e! meaning, which we found to be helpful in interpreting some of our data He proposes that the purpose or intention ofa cultural product is the meaning and further that meaning is a kind of ' cultural intention' in a supra-individual fashion. Sense is the interpretation one makes of the meaning He argues further that language is a special kind of cultural product. We also found the work on activity theory of Kathryn Crawford (1996) to be relevant. She highlights how activity denotes personal (or group) involvement, intent and commitment that is not reflected in the usual meanings of t he word in English. In building upon Vygotsky's work, Leont'ev, Davydov and others made clear distinctions between conscious actions and relatively unconscious and automated operations. Operations are seen as habits and automated procedures that are carried without conscious intellectual dl'm!

Methodology

Data was collected by means of the video recording of a series of one-hour sessions With four groups of students during March 1995. The students involved were drawn fi'om courses ill Education Engineering. The groups had two or three sessions each.

A series of tasks was devised which were designed to get the students talking toget her about their understanding of algebra. For example the first activity involved 'Algebraic Pairs' In this activity each group of two students is given a set of cards with a pair of algebraic expressions on each. The task is to decide of the two expressions are always, sometimes or never equal. Another activity was to ask them to explain what they understood by mathematical words such as expression, equation, function, variable etc. The sessions were carried out in a small TV studio.

The initial data analysis involved the three researchers simply viewing the video tapes and discussing reactions and questions arising. Following the tape transcription this process was repeated with the transcripts. Our discussions were further informed by our ongoing reading. We also held two internal university research seminars during this period.

In this paper we have chosen one particular section of the transcript which we found to be particularly rich but also very challenging to us to make sense of in terms of the sta 11ing point of our study i.e. how an focus on language and meaning can assist students in reconstructing their algebraic knowledge.

Data Analysis and Discussion

This particular section of the interaction took place at the end of the first session ""ith the:2 Year BEd students. They had been working on the Algebraic Pairs activities for the first palt of the session and then had spent the latter part in a discussion of mathematical terms such as expression, equation, function, variable etc. As the session was almost complete, the researcher provided the opportunity for any questions, reactions or general discussion. The result was an extensive and articulate series of responses from one student in particular - Anthony (AG) Anthony IS a mature student who had previously worked in industry as an engineer.

1. **BH** OK, I was going to think about further activity but seeing as we've only five minutes left, I
2.think we'll end. Unless, are there any particular things that struck you as we've been talking, that 3. you want to return to, words which conjure up ...
4. AG It's obvious as we start talking about maths, we start talking about functions, some people
5. have got a clearer view; that my image I realise now, when I'm teaching, I tend to opt for, I like
6. to see it as, that y = some function, it could be a=3b plus something. I keep returning to y =
7. some function ofx and ifI saw it in a textbook for example that 2(x+3) my automatic reaction
8.would be to write y=(2x+3) before I give it to the children to do. y=2(x+3) 9. **BH** What would you be thinking of asking them to do next')
10. AG I'd be asking them to multiply the brackets out to give me a y=2x+6 or asking them to 11. substitute a value ofx and tell me what y is because that on its own as a function - 2x+3. J12. suppose to me it isjust floating about in mid-air with no relationship to anything II'S totally 13. intangible, what is it? what's it for? So ifI ask them to multiply that bracket out I got :2x+ 3
14. before, now I've got 2x+6, still doesn't lead to anything, doesn't mean anything doesn't tell me IS. what it's from or where it's from - so my automatic reaction is to put the y= in Otherwise you've 16. got that floating about and that is a function, then you've got function. To me. what is a function, 17. where does it come from, where does it come from?
18. **BH** Y ou'd be happy to relate it to y. What would that mean then for you')
19. AG There's a missing number y and a missing number x and if we put any valul' in It)r y, or any

20. value for x .. .If we can find a value for y then we can find a value for x and if vou get into a
21. quadratic there'd be two answers for y, so actually you're using something to solve a problem.
22.**BH** Just taking that, say it was y=2x squared plus 6 times something ... You said tvvo values. 23. AG Again, as soon as you get an x squared, I tend to think that it's probably going to be two 24. answers. Depends on ...

25. AG I'm coming from a realistic point of view in that I've got a specific problem of trying to find
26. out what this value ofy is and in doing that I've made an equation in order to solve my problem
27. and in trying to solve my problem I might find that there are two values of the '\.
28. **BH** Say we had that? What about ifI said y was minus 10')
29. AG Minus IO? Then there might not be a solution to it .... no real solution. No solution to my real
30. world. This idea of no real solutions - you've gone into a hypothetical world Y ou\ e gone out of a
31. real life situation. From my experience, in my situation, you've gone out of a real life situation,
32. you're going back to a hypothetical situation. You're going right full back in circles to functions,
33. that's something hypothetical - that's floating about, not related to anything or solVing anything. It's
34. not come from any real life situation, it's just a function, it's not related to anything else I think 35. that's why I have difficulty in seeing where it's coming from.

An initial analysis of this section suggested a number of links with the background literature and theoretical framework previously outlined. In order to help the reader make sense of tile transcript, it is worth emphasising at the outset that Anthony does not distinguish between the terms function and expression. In fact he refers to 2(x+3) as a function rather than as an expression at line 7. In relation to activity theory there are a number of references to a lack of purpose when dealing with functions. For example at line 13, Anthony asks 'what is it') what's it for')' and at line 14 says that it 'still doesn't lead to anything' and goes further to say that it 'doesn't <u>mean</u> anything' This statement fits with Erik Schultz's interpretation of meaning as the 'purpose or intentic)Il of the cultural product' which in this case is the word 'function'. Anthony's description ,dso suggests that he is working operationally for much of the time e.g. at lines 7/8. he says that 'my automatic reaction would be to write y=(2x+ 3)' and also at line 15 he says that 'my automatil' reaction would be to put the y= in'. His comments also suggest a lack of appreciation of the structural properties of equations e.g. at lines 10111 he would 'be asking them (the children) to ... substitute a value ofx and tell me what y is' This suggests a view, consistent with the work of Carolyn Kieran, o1' 'the right hand side as the answer'. His comments at line 19 'There's a missing number y and a missing number x' suggest that he has not made the transition, in Anna Sfard's terms, from the 'stnlctural' algebra of 'a fixed value' to the 'functional' value of a 'variable'. It seems from Anthony's C\ll11l1lents that he sees the purpose of an equation as being to find a missing number and not to express a relationship. In Carolyn Kieran's terms, the equality relationship is not fully recognised i.e. the equation as a balance between right and left hand sides and as a structure to be operated on symmetrically

To an extent these observations are typical of many students although they were surprising to the researchers, as Anthony was seen to be a mathematically capable, though not strong, student. However much of what Anthony had to say was left untouched by this analysis and we were left with a sense of the inadequacy of the various theoretical frames, through which we had viewed our data, to account adequately for what Anthony had to say. It seemed that there was evidence of resistance to 're-Iearn' algebra on Anthony's part and much that was being said about his sense of identity and also his view of the nature of mathematics. None of this seemed to have been addressed in our first readings of the data. As a result of wider discussions with colleagues we decided to look to social practice theory for a 'wide(r) angle lens' (Robert Dengate and Stephen Lerman, 1995) through which to view our data. In particular we turned to the work ofJean Lave and Etienne Wenger (1991) and that ofJean Lave (1996).

Jean Lave and Etienne Wenger stress the essentially social character oflearning and propose learning to be an aspect of a process of participation in socially situated communities of practice They discuss the notion of Legitimate Peripheral Participation (LPP) which describes the particular mode of engagement of a learner in a new community of practice, whose level of pal1icipation is at first legitimately peripheral in the practice of the expert. The move from peripheral part icipation to full participation is seen as a dynamic process, characterised by changing levels of participation. Writing in 1996, Jean Lave describes the direction of movement as a *Ie/os* and gives the example of 'becoming a respected, practising participant among other tailors or lawyers, becoming so imbued with the practice that masters become part of the everyday life of the Alley or the l1losque for other participants and others in their turn become part of their practice' She proposes that this might form the basis of 'a reasonable definition of what it means to construct identities in pracl ice'

Returning to the analysis of the transcript, it seems that there is considerable resist,lI1ce on Anthony's part to re-construct his view of algebra. His view ofa function is that 'it is totally Intangihle' (I 13) and 'with no relation to anything' (113). It is 'floating about in mid-air' (I 12), withoul meaning e.g. 'what is it?' or purpose 'what's it for?' (113). It seems that Anthony's view of mathematics is only meaningful if 'you're using something to solve a problem' (121). Having a problem to solve is real e.g. 'I'm coming from a realistic point of view' (125) and equations are simply tools to solve 'my problem' (127) e.g. 'I've made an equation to solve my problem' (125). In formulating his views on the nature of mathematics, Anthony also seems to be saying signiticant things about hi~ own sense of identity. His background is that of an engineer working in industry over many years and his path into Higher Education and teacher training would have been via vocational routes Ant hony seems to be calling on his previous experience (as expert) in this particular community of practice and also on his developing expel1ise in the practice of 'school teacher' to emphasise his identity as a part of the 'real world' e.g. 'my experience, my situation' (I 31). This contrasts with his view of the community of practice of mathematicians, as exemplified by the researcher, who inhabits' a hypot hetical world' (130) and who has departed from the real world e.g. 'you've gone out of a real lite situation' (I 31). He stresses his view that the researcher/mathematician is going nowhere e.g. 'Y ou're going right full back in circles to functions, that's something hypothetical - that's tloating about, not related to anything or solving anything. It's not come from any real life situation, it's just a function, it's not related to anything else.' (131-35). However he does seem to express some sympathy and desire for a greater level of participation in the practice of being a mathematician when he says' I think that's why I have difficulty in seeing where it's coming from.' (1 35) This also seems to reflect his peripheral participation in this particular community of practice.

It seems that our interest at the outset of this study, in language and meaning, has given us a picture of some of the ways in which our students are working on re-Iearning algebra. However it has also revealed much more - a complex set of phenomena and questions with which to revisit both our data analysis and also the ongoing development of our own practice.

Acknowledgement

We would like to thank colleagues who attended seminars of the Mathematics Education Research Group and the Learning and Teaching Institute at Sheffield Hallam University during 1996 and also participants at the session at the BSRLM Conference in November 1996 for their contributions to the development of our thinking about this study. In particular we would like to thank Protessor Peter Ashworth and Mick Nott for their insights.

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SOME PROBLEMS IN RESEARCH ON MATHEMATICS TEACHING AND LEARNING FROM A SOCIO-CULTURAL APPROACH

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It is not my intention in this paper to argue for a socio-cultural perspective on mathematics education (see Lerman, 1996) but to examine some of the problems that one faces in research from that perspective. The international group for sociocultural research held its second meeting in Geneva earlier this year (1996) and there is a growing body of research from that group and from its members. In the UK in mathematics education I believe that we have little experience (with some notable exceptions, e.g. Solomon 1989; Nunes & Bryant 1996) and my intention here is to open a discussion about the problems of designing and carrying out sociocultural research. By 'socio-cultural' I am referring to theories which argue that social and cultural forces are *constitutive* of human consciousness not merely *causative* (Smith, 1993, p. 128). In particular, but not exclusively, I will refer to the work of V ygotsky and followers when outlining the theoretical issues that sociocultural research attempts to address in mathematics teaching and learning. It is perhaps more appropriate to describe Vygotskian research as historical-cultural rather than socio-cultural in order to emphasise phylogenesis: "The fossilized form is the end of the thread that ties the present to the past" (Vygotsky 1978, p. 64).

1. Dynamic research on learning

Learning is a constantly shifting, unending process. Vygotsky defined the zone of proximal development as the gap between what a child can do on her or his own and what she or he can do with, for example, a teacher. In order to gain a picture of what a child knows one has to take account of this. The problem is complicated by the fact that learning is different at different times and in different forms. The learning activity constitutes the zone of proximal development, the zone is not something pre-existing, attached to each child like a force-field or like a framework (a scaffolding) to lead a child from what she knows to what the teacher wants her to know (Meira & Lerman, forthcoming). Therefore any test or evaluation will provide just a snapshot of part of what children know in that/those context(s).
What is needed is some kind of dynamic research methodology which allows us to look at learning as it takes place and looks towards its future.

2. Situated meanings

Mathematical meanings are situated in practices. Contexts used in the mathematics classroom or in textbooks, or mathematical knowledge (as seen by mathematicians) in outside school social practices, particularly work ones, drawn into the classroom create particular problems for teachers and for research (Boaler, 1996). Further, as Evans's research has shown (1994), contexts call up different practices for people and meanings can shift along chains of signification, they can break away from where they begin and certainly from what is intended by the teacher. They put what the teacher intends 'at risk'. Thus students can perform differently in different situations. We need to take account of contexts, of transfer into the classroom and from school mathematics to outside practices, and of shifts of meanings, when researching mathematics teaching and learning.

3. Multiple voices

Vygotsky's formulation of human consciousness as appearing first on the social plane and only afterwards on the individual plane roots the individual's consciousness in cultural domains. But these domains are multiple, including gender, ethnicity and class. At different times and in different situations, people will feel themselves to have voice or to be denied voice. At the micro-level of children working together on mathematical tasks, this structures the interactions and knowledge construction of the group (e.g. Brodie, 1995). In research (and in teaching) we would want to enable powerlessness and powerfulness to be articulated and to take account of it.

4. Reflexivity

It is not new to say that the researcher inevitably affects what she or he is studying. Not only can one not be a fly on the wall, chaos theory implies that the fly might cause a storm on the other side of the world! There are yet more levels of reflexivity than this. As a researcher I am trying to understand what learning is, that is I am trying to learn about the situation I am observing, which is a situation in which children are trying to learn mathematics and the teacher is trying to learn about teaching. What is more, the researcher's theoretical assumptions about learning frame the perspective on the process in which she or he is involved.

5. Individual/social dialectic

Vygotsky called one of his books "Language and Thought", emphasising the dialectic of language, which is firstly social, and thought which is the individual's (though no less social in origins). Any learning activity is constituted by the actors and their scripts and particularly the teacher and her or his mathematical script, in the social structure of the classroom. How can research take account of all that? We might usefully say that focusing on the individual foregrounds thought, whereby we must take account of the background, and similarly focusing on the social setting foregrounds language and communication, but we must take account of the individuals.

6. Semiotic mediation of cultural tools

Vygotsky drew on the Marxist notion of material tools transforming the world and people to talk about cultural tools. These too construct the world for people and people are constructed through them, they mediate consciousness. The transformation is mutual and dialectic. The mediation of cultural tools has been studies throught the effects of sign systems (e.g. Luria, 1976; Nunes & Bryant 1996) and the way they structure and regulate thought. Another approach has been to draw on the semiotics of Peirce, together with a theoretical formulation of development from V ygotsky, to build a Developmental Semiotics to study meanings and their growth along lines of increasing sign-sign foregrounding in mathematics (Vile, 1996). The notion of cultural tools emphasises the role of the teacher who represents culture in the classroom, specifically in this instance mathematical culture. We might wish to use long-term studies in order to examine the effect of the mediation of cultural tools.

7. Spontaneous and scientific concepts

It is ironic that Vygotsky's method, offering a view of consciousness as constituted in cultural and social practices, should have opened up the possibility of multiple voices when at his time (post-October revolution) and in his place (Russia) he saw development as being from pre-literate to literate, from backward to advanced culture, as a single line of social progress. It is ironic but it is not surprising, given his method. Interpreting his spontaneous/scientific distinction today, the task for researchers (and, again, for teachers) is to recognise the differences and work with them. Contrary to other formulations, for Vygotsky spontaneous concepts don't die, and are not simply subsumed into more advanced concepts, they are raised to a level of consciousness and scientific concepts made concrete so that they confront each other. An example from teacher education: in an attempt to engage student teachers, in the final year of their course, with their still unchallenged assumptions about the role of the teacher, Crawford & Deer (1993) devised an activity in which the students had to work in groups to develop a programme of mathematics which was centred on the children's environment, rather than a prescribed syllabus. The students found this very hard and experienced: "initial ecstasy, shock of recognition, crisis, realism and commitment" (p. 116). The outcome was at least a recognition by the students of having a wider range of skills upon which to draw and in many cases new-found confidence in their ability to create "a very different learning environment ... from the one that they had experienced themselves" (p. 118). Elsewhere Crawford writes:

> The course was designed to create a "zone of proximal development" for student teachers as a way of expanding their knowledge of the dialectic process of teaching and learning through conscious experience of the process. They were engaged in a learning activity. (Crawford, 1994 p. 6)

Another perspective, perhaps complementary, is that of splitting: notions of a half, as in "Your half is bigger than mine" are confronted with the mathematical notion of a half, and these two meanings are seperately contextualised. Fischbein's work (1991) similarly points out that intuitions (a 'six' occurs less often than other numbers when a die is thrown) exist side by side with theoretical mathematical knowledge. Research needs to uncover the spontaneous concepts in examining the learning of scientific (mathematical) concepts.

Research

I have listed and discussed briefly elements that might frame methodology. One wants to incorporate all these, and perhaps other, elements when researching from a socio-cultural perspective. But one can't do it all.

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MA THEMA TICS IN THE PRACTICE OF VOCATIONAL SCIENCE

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As a window onto the mathematical practices of science students, a project working alongside people studying vocational science courses (GNVQ Advanced) is currently in progress. Through classroom observation, analysis of course materials, individual interviews and diagnostic tests, a picture of the students work with mathematics-inscience is emerging. The practice of converting between units of measurement has been analysed in depth. Converting is a critical aspect of science, drawing on several mathematical ideas. In this paper we present a summary of our analysis of students converting practices in a test situation, and describe an episode of converting in chemistry.

Introduction

In England and Wales, increasing numbers of 16-19 year old students are choosing to study new vocational qualifications as an alternative to the more academic' A' levels. The qualifications GNVQSl - are a competence-based qualification covering areas such as Leisure and Tourism, Manufacturing and Business Studies and more recently, Science. All students studying for these vocational qualifications must cover the core skills of communication, information technology and application of number as well as their subject studies. Further Education colleges (who are the main providers of vocational courses) differ in their approaches to the integration, or otherwise, of the core skills into the main subject (FEDA, 1995). Traditionally, mathematics has been taught as a separate subject to science students, but with examples often drawn from science. One view influencing teaching of these vocational courses centres around the idea of teaching mathematics in science contexts, sometimes within science lessons. This assumes that if mathematics is taught in this way, students will be more motivated to learn it and more readily able to use it, in science.

We are currently engaged in a project working with two groups of vocational science students with the aims of:-

- characterising the mathematical practices of these science students,
- investigating the advantages and disadvantages of the GNVQ approach to teaching mathematics in context,
- investigating the role of computer-based activities in the development of mathematical competencies the students.

At present, we are studying which mathematical practices the students engage in and how these practices are influenced by the ongoing science activity. From initial analyses of the GNVQ Science 'curriculum', a diagnostic test taken by the students and from extensive observations of classroom

[1] GNVQs can be studied at Foundation, Intermediate and Advanced level. An Advanced level GNVQ is equivalent to two 'A' levels.

interactions, several mathematical ideas were identified as pivotal. We started our analysis of mathematical practice with an in-depth study of the area of **converting between units of measurement,** an area where mathematical and scientific ideas are intimately related.

The project is influenced by the work of Vygotsky, Wertsch and Lave. We take a socio-cultural approach to mind, which means we attempt not to separate mental functioning (or action) from the contexts in which this action takes place. The 'agent-acting-with-mediational-means' is taken as our unit of analysis and as Wertsch (1991) has pointed out, "mediational means shape the action in essential ways". In our work, mediational means include laboratory equipment, calculators, graphs, algebraic representations etc. We also draw upon Lave's idea of 'structuring resources' (1988) and take the view that 'mathematics' can give structure to and be structured by, other ongoing activities.

Methodology

The project is being conducted with groups of GNVQ Science students at two Further Education colleges in different English cities.

The first phase ofthe project involved the design and administration of a 'diagnostic' test, as one angle on the mathematical practices students brought to the science course. The test covered several mathematical areas including, ratio and proportion, algebra, trigonometry and converting between units. The test design drew on GCSE Maths, GNVQ application of number criteria and knowledge of the critical mathematical ideas used in science. Questions from past examination papers (GCSE Maths and Science, Scottish '0' grade) and created questions, were presented in a mixture of science and more abstract contexts. Some conversion-related questions were;

A) Circle the larger quantity,

1×10^{-3} kg or 10g

B) Convert the following quantities;

a) 350g into kilograms

d) $5^1/_2$ hours into seconds

e) i) $45cm^2$ into $metres^2$

C) A ball bearing has a mass of 0.44 pounds. Calculate the mass of the ball bearing in kilograms. (1 kg = 2.2 pounds)

D) Write a formula to convert, grams G into kilograms K

K=

Follow-up interviews were conducted to elicit students own explanations for their ways of working on these questions. Phase two of the project involved extensive observation of the vocational science classrooms. Both 'theory' and 'practical' sessions were observed and student's notes and assignment work were also collected and photocopied for analysis. Phase 3 of the project, which will not be discussed here, concerns the design and evaluation of teaching scenarios aimed at supporting students in the development of particular mathematical ideas.

Characterising student's conversion practices

The topic of converting between units of measurement was analysed in depth using all the data sources available. Converting was chosen as the first area to study because of its critical importance in science practice in general and also because the classroom observations and test data showed converting to be of interest in terms of opportunities for studying mathematics-in-practice. Conversions are an integral part of scientific practice for many reasons, for example,

- to allow for ease of data processing
- to enable comparison and standardisation
- to support understanding of physical quantities and processes

The importance of converting arises from the integral role of measurement in science. Often measurements are taken which need to be transformed in some way, for the reasons above amongst others. The role of converting has been recognised as important within the framework of GNVQs and several references are made in the application of number criteria studied by all GNVQ students, not just science students. Thus it is crucial that the science students become competent in this area.

Diagnostic Test

Several important points emerged from analysis of the test data and follow-up interviews. We found that the students approached the solution of conversion problems in many different ways. For example, when asked to convert $5^1/_2$ hours into seconds, some students converted the '5' and the ,1/2, separately, then added the result, whist others worked with the $5^1/_2$ as a whole number. Although both methods generally led students to correct responses, we conjecture that working with parts rather than the whole mitigates against developing any generalised notion of a conversion relationship.

The majority of students were successful at converting single-step metric quantities, for example, 200mI into litres. However, the majority of students were unsuccessful at conversions involving standard form notation and units with indices, for example the questions, *convert* 1 *x* 1 *(jm* into millimetres, and *convert 45cm^2 into m^2*• The most common response to the latter question being $0.45m^2$, that is carrying out the calculation 45 + 100 and not paying attention to the unit dimension. The majority of students also had difficulty in expressing conversion calculations formally, for example, *write an expression to convert millilitres Minto litres L,* although the performance of a specific conversion immediately preceding a request for formalisation did support some students in the formalisation process. Once generated, formal expressions were used by a few students as a resource in the calculation of further specific conversions.

Throughout the test and interviews, it was noted that the students used calculators extensively, and in particular ways. For example, when converting, say 200mI into litres, many students would both multiply and divide the 200 by 1000, and then decide which answer to accept and write down. Access to the calculator allows a student to work in this way but in addition, tacit understandings of qualitative relationships are also drawn upon as resources in the students problem solving processes.

Conversions in classroom practice

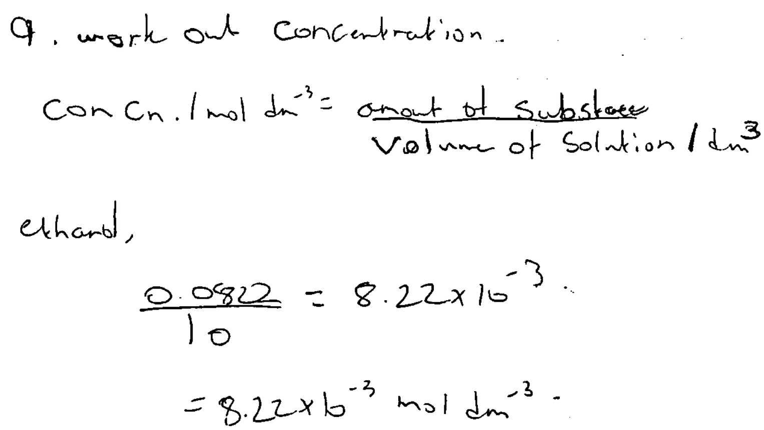
The practice of conversions was found to be prevalent in the student's classroom and assignment work. In many ways the students were experiencing converting as a practising scientist would, that is, the science situations encountered created the need for students to engage in conversions due to the large amount of measurement activity and data processing required. Conversions in the student's science experience could also be characterised as *embedded* and *implicit.* That is, the practice of converting was usually a minor element in a much larger, complex science problem and often the actual converting between units was not explicitly drawn attention to. Some of these ideas are illustrated in the following accounts of conversions-in-practice at one particular college. The chemistry lessons at this college are often divided into a 'theory' session, held in a normal classroom, and a 'practical' session where the students move into a chemistry laboratory to conduct experiments. In one 'theory' session the students worked on three questions, on a photocopied hand-out, based around the topic of 'equilibrium constant'.

> The students had laboured over the first two questions. They went on to question 3) on the hand-out. The tutor had pointed out that this question was very relevant to the practical work they were about to do. The students read through the question and the tutor asked them questions relating to the science of the situation, for example, "what is the titration doing ?" He wrote on the board the words,
>
> Amount of NaOH =
> reacted (mol)
>
> He then stated he was fairly sure that they wouldn't 'get this', even though they had done it on previous occasions. He asked "how do we work out how many moles of NaOH have been reacted ?" One student called out, "divide one by the other" to which the tutor replied "no". He continued to write on the board, completing the expression,
>
> $$\underset{\text{reacted (mol)}}{\text{Amount of NaOH}} = \frac{\text{vol. soln}}{1000} \times \underset{(\text{mol dm}^{-3})}{\text{conc}^{\text{n}}}$$
>
> **The tutor asked the students to "work this out".**

The chemistry question being attempted was a complex, multi-step chemistry problem which the tutor guided the students through. The formal expression the tutor provided contained an *implicit* conversion from cm^3 to dm^3 signified by the division by 1000'. The expression was written on the board, in it's entirety, and with no accompanying explanation. The students substituted in the values given in the problem statement, calculating an 'amount'. The expression is somewhat complex and the direct question, "how do we work out how many moles of NaOH have been reacted ?" did not seem to offer students any support in reaching it. So, the tutor tells the students the expression they need, that is, he provides the information and structure, and just asks them to carry out the arithmetic. This may be because the expression is only a small part of the larger, important, chemistry problem and in some way he wants to de-emphasise the formal aspect.

The student assignment which followed on from this chemistry lesson was to write up an experiment on 'Determination of Equilibrium Constant for an Esterification Reaction' (an 'A' level textbook problem). The evaluation of equilibrium constant from their experimental data involved a series of calculations which students carried out by following a set of questions similar to those worked on in the theory lesson. The series of ten calculations progressively led to a value for equilibrium constant, using the students own data,

> Heidi worked through the series of calculations and successfully answered the questions, up to number 9 (out of 10). She correctly recorded the expression to find 'concentration' (concentration = amount / volume) and she also recorded the correct units for each quantity. However when carrying out the calculation, she did not account for the fact that her value for 'volume' was measured in cm^3 but the expression is valid only for volumes in dm^3• Consequently she substituted and evaluated an incorrect expression, obtaining concentration values which were incorrect by three orders of magnitude.



Heidi had taken the appropriate expression, substituted her own calculated value for 'amount' and the given value for 'volume' but had neglected the conversion aspect of the problem. The complexity of the chemistry situation was such that it was difficult for the student to keep track of all aspects of the problem. Indeed, the series of calculations involved in the analysis of this particular experiment was very complex and it is unlikely that tutors would be able to keep track of such errors in an assignment either.

The expression Heidi used is the 'same' (in a rearranged form) as one she correctly used earlier in the same assignment, and was observed to use during classroom observations. In both the assignment and classroom the expression was presented in the following form,

$$\text{Amount (e.g. of NaOH used)} = \frac{\text{vol cm}^3}{1000} \times \text{conc}^n \text{ mol dm}^{-3}$$

That is, the conversion from cm^3 --> dm^3 was included, but presented in an implicit way. Heidi had no difficulty in her assignment in using the above expression to find 'amount', but when evaluating 'concentration' from the expression, neglected to include the conversion relationship.

The complex nature of chemistry calculations, and the implicit use of conversion factors, is also found in the text book which the students use (Nuffield, 1995), demonstrating that these features are aspects of the science culture in which the student is acting.

Summary

An analysis of responses to conversion questions presented in a test format, and students verbal explanations of their solutions, demonstrated that they made sense of conversion problems in many different ways. The vast majority of students were successful at straightforward metric conversions but were not successful with questions involving standard form notation nor when the units were of dimension >1 (e.g. cm^2). They often drew on qualitative strategies, along with calculator feedback, to make decisions about the quantitative problems.

Converting was observed to be prevalent in GNVQ Science classroom practice, however conversions were usually embedded in complex scientific situations and were often implicit. These points may contribute to the difficulties students were seen to face when trying to make sense of conversions throughout their science work. In the diagnostic test students were usually told to "convert" whereas in the science situation they must realise this for themselves. This recognition comes through an understanding of the science of the problem, not just the mechanics of converting.

Some students who had little success in the diagnostic test were seen to work appropriately within the context of their science work, indicating that the science practice may structure mathematical ideas in ways which do support students. However some students who were successful in the test seemed to have no access to resources to draw on in the science situation, or else utilised resources inappropriately. The challenge of the next stage of the project is to design and evaluate teaching scenarios that aim to support students in appropriating resources, fit for the purpose of their science practice.

Acknowledgements

This project is funded by The Leverhulme Trust.

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THE ROLE OF NUMBER SENSE IN CHILDREN'S ESTIMATING ABILITY

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This paper presented findings of a study looking at the comparative and combined effects of age and number-sense on children 's ability to estimate measures. While evidence was found for a developmental effect of age on children 's number-sense, no such effect was found for the ability to estimate either length or area. However, childrens' ability to use and perceive number relations, together with an understanding of the relative magnitudes of larger numbers, were found to have a significant influence on their ability to estimate area.

Introduction

The influence of numerical understanding on the ability to estimate has been assumed, observed or remarked upon anecdotally in most studies of estimating ability (eg.Carpenter et aI, 1976; Siegel et aI, 1982; Forrester et al, 1990). However, while knowledge of any domain is clearly an important prerequisite for effective estimation within that domain (Brown & Siegler, 1993), the precise nature of the relationship between number knowledge and estimation remains unclear.

Recent discussions of numerical ability have emphasised the role of 'number-sense'; a term used to encapsulate an holistic concept of quantitative intuition, or a 'feel' for numbers and their interrelationships (Sowder, 1992; Markovits & Sowder, 1994). Conceptually, number-sense has been taken to include the recognition of the relative magnitudes of numbers, the effects of operating on numbers, and the development of benchmark referents for quantities and measures (Sowder, ibid). It has been operationally defined in such terms as the ability to use numbers *flexibly* when computing, estimating, judging number magnitude, or judging the reasonableness of results; the ability to *move easily* between different number representations; and the ability to *relate* numbers, symbols and operations (Markovits and Sowder, ibid). The emphasis here is of a non-reductionistic kind, focussing upon the creative *use* and *perception* of numerical relationships, rather than accumulated knowledge or isolated skills.

By far the most research on estimation has focussed on computational estimation, as opposed to the estimation of measures. However, many activities frequently include elements of each, and both have been noted as sharing certain mathematical processes in common. For example, both necessitate some form of decomposition / recomposition, and/or the use of benchmark values (Bright, 1976; Siegel et aI, ibid), in addition to such processes as counting, visualising, and approximating (Ainley, 1991). Following previous work (Forrester et aI, 1990; Forrester & Shire,1994), the present study aimed to clarify the developmental relationship between number sense and estimation of measures in young children. Specifically, our aims were to identify (a) the developmental effects of age on both number-sense and estimating ability, and (b) the relative influence of age and number-sense on children's ability to estimate measures.

Method

A total of 62 primary school children (38 m; 24 f) ranging in age from 6-11 yrs, completed all tasks over a period of seven months (two school terms). All tasks were produced using Macromedia

Director multimedia authoring software, and presented on a small laptop computer. Data were recorded directly by the computer, in a form suitable for subsequent analysis.

Assessment of Number-Sense

Number sense was assessed using three tasks, each focussing on one aspect of number sense (although necessarily including certain elements of the others).

(i) Mental Computation

This task required each child to solve arithmetic problems, randomly generated within specified constraints corresponding to a given level of difficulty, and presented in the form of thought bubbles emerging from an animated head. The successive difficulty of each problem depended upon a running score, which itself increased or decreased according to the child's responses. This feedback loop thus caused the score to eventually level out at a number taken to be a measure of the child's mental computational ability. A return to the same score three times was taken as the criterion for stopping the assessment. Rexibility was built into the program such that the assessor was free to choose an appropriate entry level, or to jump between levels at any point during the assessment. working level of ability.

(ii) Understanding of Relative Number Magnitude

Children were assessed on their understanding of the relative magnitude of numbers firstly from zero to one hundred, and secondly from zero to one thousand. The display presented a horizontal line from which ten 'strings' were vertically suspended, together with ten 'oranges' marked with selected numbers between, and including, zero and a hundred, or zero and a thousand. The task consisted firstly, in placing the oranges on the strings in correct numeric order from left to right, and secondly, in moving the strings horizontally such that the spaces between them reflected the relative magnitude of their corresponding numbers. Children were given a brief demonstration, and then asked to "move the strings along the line, so that the spaces between the strings show how much larger, or how much smaller, each number is than the other numbers". A measure of overall performance, termed the Relative Number Magnitude Index (RNMI), was then calculated on the basis of comparisons between the actual and expected positionings of each number relative to the others (unfortunately, restrictions on space prevent further elaboration here).

(Hi) Understanding of Number Relations

As a capacity to learn, or to perceive and use number relationships, is implicit in the idea of number-sense, this assessment drew on Vygotsky's notion of a 'zone of proximal development' (ZPD) (Vygotsky, 1978) in an attempt to take this into account. Conceptually, the ZPD refers to the 'distance' between what a child is able to do independently in some domain and what s/he is able to do in collaboration with a more experienced other. The former may be regarded as the 'actual' developmental level, and the latter as the 'potential' developmental level. In practical terms, a ZPD assessment typically makes use of some form of graded-prompt procedure (Brown & Ferrara, 1985), in which a child's learning is assessed in terms of the number of standardised prompts needed to complete a task. Children's understanding of number relations was here assessed in a similar manner.

Each child was first presented with the sum" 18 + 36", and asked to think of as many ways as they could of how they might solve it 'in their head'. There were three target strategies; an 'add on / take off' strategy, one decompositionlrecomposition strategy based on tens and units, and another based on factors. Where it was clear that the child was simply relying on a mental image of a written algorithm, this was recorded seperately. All other spontaneous strategies were recorded under one global category (although in practice, the only other strategy suggested was 'counting on').

If children suggested, and successfully used, a target strategy spontaneously without prompting, they scored four points. After all unprompted suggestions had been made, the assessor began the prompt sequence for each of the three target strategies in turn. (For reasons of space, the prompt schedule has not been included in this report). Whenever a child was able to use a prompt to successfully solve - and/or explain how to solve - the problem, the prompt sequence for that strategy was stopped and the corresponding score recorded. The fifth prompt in each sequence amounted to telling the child how to do it, and scored zero points. The total score across all three target strategies was taken as the overall measure of task performance on that session.

Children then repeated the assessment with a new problem, "24 + 48", and the overall mean score across both sessions taken as a measure of their ability to use and perceive number relations.

Assessment of Estimating Ability

Children were assessed on their estimation of both length and area within each of two distinct contextual frames. One of these presented the task within a stereotypical mathematics 'textbook' format; the other within a 'story' format. The story itself was designed so as to appeal to children across a wide age range, and concerned the plight of some ladybirds amidst a deluge

(i) **Length**

The 'story' task was preceded by a short animated sequence (including sound), during which a twig floated downstream, coming to rest at a leaf on which several ladybirds were stranded. The ladybirds then walked onto the twig one by one, forming a line from one end of the twig to the other, after which the twig floated away again to safety. Children then completed each of six trials, during which twigs of various lengths floated downstream towards the ladybirds, whose size also varied from one trial to the next. Each trial required the children to estimate how many ladybirds of the given size would be able to fit along the twig, and so escape to safety. Each trial of the corresponding 'textbook' task simply required children to estimate the length of a horizontal line in the centre of the screen, using a shorter line at the bottom of the screen as a unit measure

(ii) **Area**

As with length, the area 'story' task was again preceded by a short animated sequence, here consisting of ladybirds crawling around in a puddle during a heavy rainfall while attempting to climb onto a floating leaf. Following this, children then completed six trials in which they were presented with leaves of various sizes, and required to estimate how many ladybirds of a given size would be able to fit onto each leaf 'leaving as little gaps as possible'. Each trial of the corresponding 'textbook' task required children to estimate the area of a rectangle in the centre of the screen, using a smaller rectangle as a unit measure .

. All unit target ratios were the same across both 'story' and 'textbook' conditions in each domain of estimation, and the order of presentation of trials randomised across subjects.

Results

Age-Related Differences

No significant differences in estimating ability were found across age groups. However, children were significantly better overall at estimating length than they were area ($F = 153.9$, df 1; $P < 0.001$). 9-11 yr olds were significantly better at mental addition / subtraction than 6-8 yr olds ($F = 8.9$, df 5; $P < 0.001$), while 10-11 yr olds were significantly better at multiplication / division than 6-9 yr olds ($F = 14.2$, df 5; $P < 0.001$). No significant differences in mental computation ability were found within these wider age groups.

In general, understanding of relative number magnitude also improved with age ($F = 3.35$, df 5; $P = 0.01$), while children's understanding of the relative magnitude of numbers to one hundred was better overall than that of numbers to one thousand ($F = 99.88$, df 1; $P < 0.00$ 1).

While all age groups improved significantly across sessions on the ZPD task ($F = 64.05$, df 1; $P < 0.001$), 9-11 yr olds' use and perception of number relations was significantly better than that of 6-8 yr olds age ($F = 7.22$, df 5; $P < 0.001$). Again, there were no differences within these wider age groups.

In summary, then, whereas children's number-sense did show improvement with age, their estimating ability did not.

Number Sense and Estimating Ability

No correlation was found between children's estimation of length and any variable except their estimation of area. However, estimation of area correlated highly with all three measures of number-sense. Standard regression analysis showed 22% of the observed variation in children's estimation of area to be accounted for in terms of age, mental computation, understanding of number magnitude, and use and perception of number relations ($R2 = 0.22$; $F = 4.04$, df 4; $P < 0.01$). Examination of part correlations suggested the greatest influence to be due to mental computation. (Indeed, a stepwise regression accounted for 19% of the variation in estimation ability in terms of mental computation alone ($R2 = 0.19$; $F = 13.45$, df 1; $P < 0.001$)).

As would be expected by definition, all three measures of number-sense were themselves highly intercorrelated. However, whereas children's mental computation correlated highly with both their understanding of the relative magnitude of numbers to one thousand, and their use and perception of number relations, the latter two variables were less highly correlated with each other. This suggested that understanding of number magnitude and number relations might usefully be considered sub-components of mental computation ability. On omitting mental computation from the analysis, subsequent standard regression showed 19% of the variation in childrens' estimating ability to be accounted for in terms of their understanding of relative number magnitude to one thousand, and their use and perception of number relations ($R2 = 0.19$; $F = 3.29$, df 4; $P < 0.05$) (ie.comparable to that accounted for by mental computation alone). Examination of part correlations suggested the greatest influence to be due to their use and perception of number relations, and secondarily their understanding of magnitude. (Stepwise regression found 15% of the variation in estimating ability to be accountable for in terms of use and perception of number relations alone ($R2 = 0.15$; $F = 10.82$, df 1; $P < 0.01$».

Hence, whereas childrens' ability to use and perceive number relations and, secondarily, their understanding of relative number magnitude, appeared to influence their ability to estimate area, no such influence was found on their ability to estimate length.

Discussion

These findings offer some support for the complementary roles of benchmark knowledge (relative magnitude of numbers) and decomposition-recomposition strategies (use and perception of number relations) as aspects of number-sense influencing children's ability to estimate measures (specifically, area). However, the fact that number-sense accounts for only 19% of the observed variation in estimating ability suggests other factors must also be involved. This is further implicated by the somewhat anomolous finding that while number-sense improves with age and estimating ability improves with number-sense, estimating ability itself does not improve with age.

Given that the domain of estimation under investigation is spatial, it would seem reasonable to hypothesise a possible role for spatial imagery ability here - specifically, the ability to mentally scan and manipulate such images. Existing research in this area suggests that younger children are relatively poor at this (Kosslyn et al, 1990), and we are currently investigating the comparative roles of imagery ability and number-sense in influencing children's estimation of volume. However, our previous research (Forrester & Pike, in press) suggests that children's prior experience of learning to estimate - and what that actually means in terms of social practice in a situated classroom context may be another major, if not overriding, influence. Specifically, what children understand by 'estimating' may be highly variable and context-specific, such that their 'ability' in this area tends to be defined and embedded within particular kinds of classroom activity and relationships, rather than be seen as a general cognitive skill (in contrast, say, to 'mental arithmetic'). One consequence of this is that children tend neither to be taught, nor to invent, general strategies for making estimates.

The study also highlights a need for testing methodologies and appraisal methods that take an ability to learn into account, rather than 'controlling' for it. The ZPD graded-prompt technique used in the present study represents one possible step in that direction, and in a later study we plan to use a similar methodology for assessing estimating ability itself. This will then allow investigation of whether children's ability to learn in one domain (eg.number) may be related to their ability to learn to estimate. Such an approach both acknowledges children's prior experience (or lack of experience) of estimating, and avoids the unnecessary and misleading conceptualisation of estimation as a static 'abili ty'.

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THE VYGOTSKIAN PERSPECTIVE AND THE RADICAL VERSES THE SOCIAL CONSTRUCTIVISM DEBATE

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Despite its widespread support, constructivism's credibility is now under scrutiny. Many constructivists refer to Vygotsky as an authority to substantiate their position; however, the various interpretations of Vygotsky open an interesting element in the constructivism controversy. This paper will argue that the Vygotskian perspective is the attempt to build a psychology structured by the Marxist epistemology of theory and practice, that to know the world it is necessary to change it by our interaction: if we are to explain the mental functions of the student as a developmental process, then we must facilitate the completion of a task that the student cannot do unaided To place the constructivism controversy within the Vygotskian perspective, a parallel will be drawn between the positions taken up in the controversy and the subjective, consensus and objectivist positions that have been adopted in the philosophy of science.

1. Introduction

Despite its widespread support, constructivism has recently become under scrutiny. The constructivist controversy has raised many issues, but the central issue in the debate seems to be the question *what is knowledge and how is it constructed?* For von Glasersfeld (1894, 1995, 1996), knowledge is an individual construction that describes or refers to the individual's experience, and does not describe or refer to an objective real world. For Ernest (1994), knowledge is an individual construction in response to experiences in social contexts. However, if knowledge is subjective and not of an external world then how is the social reconciled with the individual? The following three passages highlights the issue:

The major confusion that arises from the desire to claim that knowledge is constructed by the individual but that sometimes knowledge is absorbed from culture (Cobb, et ai), or as social convention (Ernest) or through the role of the social dimension (Bauersfeld) is that as long as the individual is at the heart of the process, as the one who ascribes meaning, any social interaction is itself interpreted individually ......As long as there is a separation between the subject and the world, including other people, one has to go all the wiry with solipsism, or give it up (Lerman, 1994).

Some critics say that the emphasis on subjectivity is tantamount to solipsism, because, they seem to think, it implies that individuals are free to construct whatever realities they like; others claim that the constructivist approach is absurd, because it disregards the role of society and social interaction in the development of an individual's knowledge. Both objections are unwarranted ....(von Glasersfeld, 1995).

But since our scientific knowledge is concordant with that of others within common understandings of the world, there is no practical room for doubt that it is the common world that we understand and have knowledge of It is all very well for von Glasersfeld to be 'post-epistemological', but to insist that it is not the world that we know is more like the post-modernist cutting off the branch on which one sits. Amusing as creative writing, but not to be taken seriously or taught to children. The very discussion of constructivism relies on common understanding; it is important for consistency *and for teaching that it does not lapse into a self-contradictory absurdity comparable to that of a proselytizing solipsism* (Thomas, 1994).

Radical constructivism replaces the objective content of knowledge with subjective expenence. Social constructivism, on the other hand, reduces the objective content of knowledge to social acceptance. For example, consider the following claims by Ernest (1991):

Publication is necessary (but not sufficient) for subjective knowledge to become objective mathematical knowledge (page 43).
To identify the immutable and enduring objectivity of the objects and truths of mathematics with something as mutable and arbitrary as socially accepted knowledge does, initially, seem problematic. However we have already established that all mathematical knowledge is fallible and mutable. Thus many of the traditional attributes of objectivity, such as its enduring and immutable nature, are already dismissed. With them go many of the traditional arguments for objectivity as a super-human ideal. Following Bloor we shall adopt a necessary condition for objectivity, social acceptance, to be its sufficient condition as well (page 45).

Radical constructivism may be generalised as a form of *subjective-idealism,* whilst social constructivism may be classified as a form of *intersubjective* (or *collective)-idealism.* Both radical and social constructivists refer to some of the positions that have been adopted in the philosophy of science to establish credibility. To place the constructivist controversy in perspective, this review will refer to Chalmers (1978) distinction between the *subjective, consensus* and *objectivist* approaches in the philosophy of science.

Many radical and social constructivists refer to Vygotsky as an authority to substantiate their position (e.g. Cobb, 1996; Ernest, 1991, 1994; Fosnot, 1996; Jaworski, 1994; Smith, 1994; Steffe and Tzur, 1994; von Glasersfeld, 1995), and the main interpretation seems to be that no separation can be made between the individual and the social context within which knowledge and understanding develops. However, the next section will attempt to show that this interpretation is somewhat naive. For Vygotsky, all higher mental functions originate as social relations - but a distinction can always be made between inter- and intra-psychological functions:

An interpersonal process is transformed into an intrapersonal one. Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (in terpsych ological) , and then inside the child (intrapsychological). (Vygotsky, 1978, author's emphasis).

For Vygotsky, the link between the social and the private is the learning of scientific concepts.

Vygotsky believed that it is through learning concepts separate from the immediate and the concrete 50.

that structures are provided. It is by learning scientific concepts divorced from immediate concrete experiences that structures are supplied (by the scientific concepts) *for the upward development of the child's spontaneous concepts toward consciousness and deliberate use* (Vygotsky, 1962; page 109). This is only possible because:

In the scientific concepts that the child acquires in school, the relationship to an object is mediated from the start by some other concept. Thus the very notion of scientific concept implies a certain position in relation to other concepts, i.e., a place within a system of concepts. It is our contention that the rudiments of systematization first enter the child's mind by way of his contact with scientific concepts and are then transferred to everyday concepts, changing their psychological structure from the top down (Vygotsky, 1962; page 93).

This raises two essential points. Firstly, Vygotsky's treatment of scientific concepts (or *decontextualised concepts* in general, which would include mathematical concepts) implies an objectivist approach towards the subject matter to be taught. The objectivist approach will be discussed in section 3. Secondly, Vygotsky's relation between the private and the social is essentially one of *method* - that to understand the higher mental functions of an individual as a developmental process, the teacher or researcher has to facilitate the process. This will be discussed in section 2.

2. The Social Verses The Radical

For Ernest (1994), there is a Vygotskian version of social constructivism For Lerman (1993), social constructivism is incoherent and he suggests that it should be replaced with a Vygotskian theory of mind. If much reference is made to Vygotsky as an authority, but that authority is open to interpretation, then it may help matters if an attempt is made to understand his epistemological perspective - the considerations that underlie his method of research. Vygotsky was a Marxist and attempted to build a Marxist psychology (Cole and Scribner, introduction to Vygotsky, see Vygotsky, 1978; Wertsch, 1985). The following is an attempt to understand Vygotsky but within the context that he himself constructed: a psychology structured by the Marxist epistemology of *theory and practice.*

The Marxist epistemology is that although we can think about the world, nevertheless we can only know the world if no separation is made between theory and practice. According to Marx: (1969):

The question whether objective {gegenstandlichej truth can be attributed to human thinking is not a question of theory, but is a practical question. In practice man must prove the truth, that is, the reality and power, the this-sidedness {Diesseitigkeitj of his thinking. The dispute over the reality or *non-reality of thinking which is isolated from practice is a purely scholastic question. (Thesis no. II,* author's emphasis).

To know the world I must interact with it, otherwise any theory I have about the world would be pure metaphysical assumption. However, by interacting with the world I also change the world - just as the anthropologist changes the behaviour of the tribe that he or she is researching. *The philosophers have only interpreted the world, in various ways; the point, however, is to change it* (Marx, 1969, *thesis no. XI,* author's emphasis). This statement is not making the ethical judgement that causing revolutions is a more worthwhile occupation than armchair theorising, it is making the epistemological point that I have to change the world in order to know it. For Lukacs (1974a):

Marx clearly defined the conditions in which a relation between theory and practice become possible. 'It is not enough that thought should seek to realise itself; reality must also strive toward thought '.

For example, to verify Snell's law for the refraction of light, a light-box must be physically constructed so that a parallel beam of light can be produced - a parallel beam of light did not exist in nature prior to its invention. A parallel beam of light is an invention, or creation, required by theory to explain refraction as a physical phenomenon. *It is precisely the alteration of nature by men, not nature as such, which is the most essential and immediate basis of human thought* (Engles, *Dialectics of Nature;* quoted in Vygotsky, 1978. Emphasis given). It is in the light of this Marxist epistemology that we can begin to understand Vygotsky. Vygotsky's approach was called the 'experimental-developmental' method *which calls for an experimenter to intervene in some developmental process in order to observe how such intervention changes it. Again the primary motivationfor doing this is to observe genetic processes: 'Our method may be called experimentaldevelopmental in the sense that it artificially provokes or creates a process of psychological development. This approach is equally appropriate to the basic aim of dynamic analysis. If we replace object analysis by process analysis, then the basic task of research obviously becomes a reconstruction of each stage in the development of the process: the process must be turned back to its initial stages (Vygotsky)* , (Wertsch,1985). In a child's zone of proximal development, ifwe are to understand the child's cognitive abilities as a *process* then we have to instigate that process by interaction with the child. It is in this sense that the 'social' cannot be separated from the 'individual'. If a child can successfully complete a task unaided, then prior knowledge of the abilities required to complete the task will enable us to say what abilities the child has. However, we would be looking at the child's abilities that have already matured in the child - we would be looking at a 'snap-shot' of the maturation process as an end-product. To understand the maturation process as a process then we would have to facilitate the child's completion of a task that she or he cannot do unaided. How a child responds to the mediation in completing a task enables us to explain the abilities of the child as they mature, rather than simply describe the abilities that have already developed.

Although all forms of thought may be considered as originating from social relations and interactions, nonetheless it would seem unlikely that Vygotsky (or Marx:) would have denied that forms of thought can also be evaluated in separation from their social roots. The truth of the theorem of Pythagoras or the validity of the Newtonian system is independent of the social relations that existed at the time of Ancient Greece or the research programme of the renaissance Gust as art can be aesthetically judged independently of the considerations of the historic and social conditions that gave rise to the art): ....*the relations between origin and validity are much more complex here than in the case of the forms of the objective spirit. Marx saw the problem clearly: 'But the difficulty does not consist in realising that Greek art and epic are bound to certain social forms of development. The difficulty is that they still give us artistic pleasure and that, in a sense, they stand out as norms and as models that cannot be equalled.* ' .... *just as it is clear that Copernican astronomy was true before Copernicus but had not been recognised as such.* (Lukacs, 1974b). In much the same way, the Vygotskian perspective considers all higher mental functions originating from the interaction between human beings - but the functions themselves can transcend the context from which they originate. A student's understanding of mathematics may be evolving such that the understanding is still specific to the examples given and the way the subject is taught. A fully evolved understanding of mathematics, at any leve~ is independent of the specific examples used, and the approach taken, by the teacher.

3. The Constructivism Controversy Placed Within the Context of Three Approaches in the Philosophy of Science.

All forms of thought originate from social relations and interactions, but the products of thought can be evaluated independently of their social origins. A position adopted can be evaluated within the context of its adoption, but the logic or validity of the position can also be evaluated independently of its origins or social context. For Lerman (1994), however, no such separation should be made:

Positions adopted carry with them much more than 'ideas'; indeed it would contradict my support for the assertion that knowledge manifests in practices were I to claim that those ideas are independent of social practices.

Lerman's position seems to be a *consensus* one: that the best theories will be those that best meet the standards and needs of an academic community.

According to Chalmers (1978), there are three approaches to the question ofthe nature of scientific knowledge: the *subjective* approach, the *consensus* approach and the *objective* approach. In the consensus view, the beliefs of individual scientists are subservient to those of the scientific community. The consensus approach is a valid one, but it is severely limited because it considers the social group the primary notion and the actual science itself: including its practice, as the secondary one. However, a particular community mayor may not practice a legitimate science; and different social set-ups have created different legitimate contributions to a single physics (e.g. Galileo's Italy and Newton's England, or the United States and the Soviet Union). For the objectivist approach, this implies the existence of physics as an autonomous practice, independent of individual or consensus opinion, that constitutes the activity involved in its development. Although the development of scientific theory is dependent on the participation of the individual scientist and of the community, nevertheless, scientific theories bear a relationship to each other and to available evidence independently of whether or not the individual or the community realises it. In the subjective approach, scientific knowledge is held by individual scientists whether it be derived from sensory experience, intuition or reasoning.

Radical constructivism is a subjective approach because it places the emphasis on individual cognition. For von Glasersfeld (1995), scientific knowledge is a question of 'viability' that employs 'fictions' , such as gravity, that are useful as a substitute for something that is inaccessible ('ontological reality'). Such fictions *can explain anything you want to explain,* and should be *recognised as tools for the rational organisation of experience and not mistaken for phenomena that are real in the sense that they themselves could be experienced.* This is an admitted Instrumentalist position (von Glasersfeld states that *Radical constructivism is uninhibitedly instrumentalist.* Page 22) that does not square with the history of science. According to Chalmers (1978), it is an embarrassment for instrumentalists that theoretical fictions can be seen almost 'directly' through electron microscopes, or be seen colliding with smoke particles in the phenomenon of Brownian motion, or that the fields of Maxwell's electromagnetic theory can be produced in a 'visible and almost tangible form' (Hertz).

According to Chalmers (1978), the product of scientific practice is the maze of theories that make up a science. At a particular historical juncture the maze of theories will constitute a *problem situation* that will have an objective, autonomous existence. Scientific theories can and often do have consequences that were unforeseen and unintended by the original proponents of the theory. These consequences exist as properties of the theory that are there to be discovered by further scientific practice. For example, when Maxwell introduced the concept of a displacement current to Faraday's concept of an electric field, he was unaware of the far reaching consequences of such a move, namely, that it predicted a new kind of phenomenon, radio waves. This consequence was not realised until two years after his death. That problem situations provide objective opportunities helps to explain the many examples of simultaneous discoveries in science, such as the law of conselVation of energy by several independent workers in the late 1840s.

5. Conclusion

By analogy with science, we may regard mathematics as an autonomous objective practice that creates problem-situations the validity of which are independent of individual cognition or social acceptance (despite its fallible and mutable nature). This is important in the teaching of mathematics from a Vygotskian perspective, that it is the teaching of decontextualised concepts that enables the facilitation of cognitive growth. If the validity of mathematical knowledge is confused with its origin, or if knowledge is emphasised with experience, then the concepts taught will no longer be decontextualised. For Vygotsky, learning precedes development and so it is by the teaching of decontextualised concepts that the student's cognitive framework comes to life. The subjective or intersubjective approach to the teaching of mathematics may result in the denial of the student's full intellectual development.

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PROBLEMATISING CONFIDENCE: IS IT A HELPFUL CONCEPT?

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Abstract
The word "confidence" was used frequently by teachers in unstructured interviews about teacher assessment of mathematics. The author analyses how it was used andfinds that different relationships are perceived between confidence and achievement. It is conjectured that different relationships lead to different kinds of teacher action or inaction. Lower usage among the secondary teachers than among primary teachers is examined and discussed, the author finding that the secondary teachers used a slightly more subject-specific vocabulary in similar circumstances, seeing links between thinking, hard work and achievement. A lack of specific description of mathematical achievement is a feature of both groups.

Context

I have been using unstructured interviews with primary and secondary teachers to find out how they make judgements about what pupils know and can do in mathematics. My raw data has been transcribed interviews which I have explored for information about the language and practices of informal teacher assessment. Interviews were typically conducted at the end of a day during which I had worked as a support teacher under the teacher's direction, and were usually in the teacher's own classroom, or some other place the teacher chose. Interviews were informal but based around some core questions, teachers being asked to elaborate or exemplify their answers in various ways. From the interview data is being developed into a model of teachers' informal assessment practices [Watson, 1995]. This paper relates to a small part of the initial analysis.

Confidence

While transcribing I noticed that the word "confidence" occurred frequently, so wondered how it was being used. I picked out occurrences of the word, in context, in the transcripts and clustered them according to similarity of usage.

Use within a community of practice can carry understood meanings and can also obscure other possible meanings, forestall analysis and influence practice. It is important to remember that the word was being used in the context of describing practices and judgements to an outsider, hence meanings were expected to be shared. I assume that teachers presented an edited version of themselves to me, and would choose language which they felt presented their practice in a fair and understandable light [Goffman,1959]. "Confidence" is worthy of attention because it is presented neither in literature on assessment nor explicitly in my interviews as a key idea in assessment of mathematics; nevertheless the word appeared in a variety of contexts during the interviews and I also noticed it in the Evaluation of Implementation of National Curriculum [Askew et al, 1993]. In my interviews it was generally used in the context of describing the mathematical potential of individual students or particular groups, while in the Evaluation it was used in the context of describing achievements under A Tl.

Taking the view that usage would indicate meaning I eventually identified five shades of meaning:

- Trust in something;
- Trust in someone specific to provide support;
- Trust that one can handle a familiar situation;

•Enterprise, in the sense of willingness to take a risk in a new situation; •
Assurance as a general personality trait.

I shall provide examples of some of these and critique their use by examining whether they serve to promote or forestall teachers' actions, or whether they are seen as unrelated to teachers' actions. Quotations are from interview transcripts.

Trust in something or someone

To go into the abstract needs them to have a lot of confidence in their own ability (MD)

Rather than re-Iabel this as "self-confidence" usage suggested trust in a separate object, property or process: something that would not let them down, like a calculator perhaps. There is a sense of the object being tried and tested and known to work well. It implies enough previous experience to know how the object usually performs. A constructivist view of how mathematics is learnt requires a certain amount of trust in the processes of learning; trust that the experiences the teacher provides will trigger some mental activity in the learner. This quotation suggests that the learner can develop that kind of trust for themselves.

I think it was a lack of her confidence in the student who taught them (OJ)

This second quotation uses "confidence" in terms of trusting some human agent, or, in this case, not trusting. There were other similar uses in which teachers spoke of having confidence in their pupils. As I did not explore this aspect through interviews with teachers I have no more to say about it, except that pupils' trust in their teachers seems intuitively to be important in learning.

Trust in a familiar situation

She's more confident with investigative work because she doesn't feel "this is matks so I can't do it" (DP)

She is quite confident with finding 2/3 *or* 3/4 *of things ... I don't think she is confident about recording everything* (DR)

He is not fully confident about writing out division sums (MD)

This meaning of confidence seems to be connected to previous successful experience in some aspect of mathematics, possibly a technical aspect. In order for the situation to become familiar it needs to have been experienced several times, for instance, a technique rehearsed several times, so that it can be recognised as one which has been done before, and the emotions connected with it are those of success. It may be a euphemism for "getting things right" as opposed to "getting things wrong".

The changeability implied here is important; in some cases, such as the last two above, it is clear what sort of experiences the teacher will provide to lead to complete confidence. Success in mathematics is seen to lead to this kind of confidence; hence the teacher, by providing situations which lead to success, can affect confidence.

There could be a problem if this in the *only* approach used to build confidence, however, as it may suggest a step-by-step, path-smoothing approach is all that should be offered to those who lack "confidence". These tender approaches may lead to performance success, but are unlikely to provide success, or confidence, in more complex mental processes.

Enterprise

You can see whether they are more confident spatially, verbally or numerically (DP)

Just the ones who are confident enough with handling numbers to have a go, and confident enough to handle explanations (DR)

.. a kind of confidence to be able to think that through and not just use the methods I showed them (FJ) *To me a mathematician is someone who is confident in tackling new ideas and using previous knowledge to work it out* (MD)

There certainly seems to be a tradition of using the word to describe the willingness to take risks, to have a go at something, to be enterprising. Some of these require specific knowledge or dispositions to be sufficiently embedded in the learner's mind that they provide safety (for instance, spatial confidence). Others require previous successful experience (for instance, the tackling of new ideas).Confidence in this sense cannot be regarded as a fixed state of mind since it is based on skill or prior experience. Teachers can make a difference to skill acquisition and successful experience of risk situations. Provision of learning situations in which a child can develop mathematical enterprise allows the teacher to refer back in future to successful past situations and reduce the perceived risks of "having a go" in future. A wareness that this kind of confidence is an issue within mathematics, rather than as a given aspect of personality, can prompt teacher action. It would be easy, however, to misread it as personal assurance and hence regard it as outside the influence of the mathematics teacher. I suggest that the causal relationship here is that success leads to confidence, rather than the other way round.

Assurance

I think mathematics is a bit of a confidence trick. I believe you can do almost anything you can believe you can do (VI)

If it was a confident child I might be prepared to take the risk and give a test(DR) She will do well as she increases in confidence (MD)

Throughout the analysis I became increasingly concerned about the abstract acquisition of confidence implied by this usage. It is as if a child either is or is not confident and the successful learning of mathematics follows, or not, accordingly. The last quotation brings this home clearly and raises the question "how is she to increase in confidence?" The answer may be "by doing well" and the teacher may indeed be looking for ways for her to succeed as a precursor to growth of confidence, but the statement implies the inverse causality. Of course there are children who are assured whatever they do. This assurance can come from successful parenting, past success, understanding of school culture and ability to demonstrate that they can fit in with expectations. Lack of assurance may be deep-rooted and unchanged by anything a teacher does.

Teaching for confidence

There is, therefore, a choice to be made. Should the teacher of mathematics accept the level of confidence the pupil demonstrates in the subject, or should she work in such a way that will boost confidence? If confidence can grow as a result of a build up of successful experiences, then it seems realistic to assume that the new confidence could be generalised to any mathematical situation, in time, rather than confined to one context (investigation, fractions or writing out division sums) which may have given rise to the success. Two teachers may have been suggesting this when they talked of personal confidence arising out of particular aspects of mathematical thinking:

His confidence has increased this year and he is beginning to think in the abstract where he couldn't before (MD)

Once she has got going and got her confidence she's quick to see how to get connections (DP)

My worry about this approach is that success may only ever related to small steps. The learner is then dependent on being offered small steps. I looked through the interviews for any voluntary mention of pupils having confidence to take large jumps or suggest links and connections in general. These are essential attributes for pupils to be independent thinkers in mathematics. Nowhere did any teacher talk about how these mental skills might be developed through teaching, although they were mentioned a few times as important, mainly in the context of A Tl, specific assessment activities or GCSE coursework.

Differences between primary and secondary teachers

From the interviews it appears that the primary and middle school teachers use "confidence" significantly more than the secondary teachers when asked about pupils' mathematical behaviour, achievement and potential. One of the primary teachers who used it more than once used it nearly thirty times in an hour! I am not, however, misled into thinking that its use implies that teachers always attempt to develop confidence in their pupils, nor that non-use implies non-awareness. I suggest the following possible explanations, intended to be non-judgemental:

- that primary teachers, who may not be specialists, use "confidence" as a catch-all and explain-all phrase in the description of learning behaviour for which they may lack a vocabulary or analysis in their community of practice. They may not understand what learning and doing mathematics entails in detail and hence may not be able to teach it, therefore using "confidence" to describe the otherwise indescribable

- that primary age children need to develop confidence, of all kinds, above all else and hence it is high on teachers' agendas
- that secondary teachers, being more confident in mathematics themselves, have more specific language available to talk of mathematical ability or mental demands
- that secondary teachers, being primarily teachers of mathematics rather than teachers of children, regard confidence as outside their purview or as a "given"
- that secondary teachers, having worked with coursework for several years, have integrated confidence-building into their work to such a degree that it no longer needs naming, having a wider knowledge, vocabulary and understanding of what it takes to be successful in mathematics

I do not intend to offer any of these as the "truth". It is interesting, however, that the word is most frequently used in the context of describing ability, learning behaviour or potential and hence its use may indicate mismatch between the non-specialist teachers' experience of mathematicalleaming, the specialist teachers' experience and the language available to describe these experiences. To shed light on these possibilities I attempted to find out what the interviewed secondary teachers had said in place of confidence. The nature of the interviews made it extremely difficult to compare context for context. I identified four possible general contexts to help me search the secondary transcripts: descriptions of individuals, descriptions of what teachers look for in children in general, decisions about setting, and descriptions of attitudes and mental states. I selected, on the basis of the meanings I had identified, what teachers *might* be talking about in these contexts where the concept of confidence *could* have been used, but was not. It was an exercise to discover what might be in the mind of secondary teachers where "confidence" might be in the mind of primary teachers[1].

Out of 9 secondary interviews analysed the following ideas emerged strongly: usefulness of specific ways of thinking; hard work; communication; previous knowledge; pupils may have to be pushed; brightness or natural ability; pupils need challenge. These ideas also appeared strongly in primary interviews, so confidence could be an additional idea, rather than a substitute. However, it was also noticeable that secondary teachers were willing to talk about what the pupils *think* when asked about individuals, showing a little more articulate awareness of mathematics. Perhaps descriptions of thinking were taking the place of descriptions of personality. Typically teachers talked about such aspects as "willingness to half-formulate ideas", "willing to think things through", " ... where she is in her thinking", "gathering thoughts", "ways of thinking" and the negatives of these were often given in

1 The reader must be aware of the layers of interpretation involved here. First there is my analysis of use, then my clustering of usages into separate meanings, then my choice of context in other transcripts, and then my matching of use to the clusterings of meaning. Every stage of this is subjective. My purpose is to raise questions about delicate judgements through examining how they are articulated in unstructured interviews about teacher assessment of mathematics.

descriptions of weaker or insecure pupils. There was, however, no mention of any specific thinking strategies, apart from being systematic, so I cannot wholeheartedly say that the secondary teachers were more fluent about a range of thinking processes

Conclusion

The idea of "confidence" was used with varying frequency, much more by primary than by secondary teachers, to describe a range of emotional aspects of doing mathematics. Broadly these are about trust, enterprise and assurance. In some uses teachers seem to believe that confidence affects mathematical success, in others that success affects confidence. Sometimes specific areas of mathematics can be identified to work on and hence build confidence; other times it is a general personality trait which is possibly not regarded as changeable within the mathematics classroom. There was recognition that taking mental leaps and risks was part of being confident and part of learning mathematics, but the teachers interviewed did not in general relate these causally.

I suggested several possible reasons for the differences in its use. These range from differences in mathematical knowledge and experience of teachers to differences in recognition and care about confidence levels of pupils. Other aspects of my research suggest that the former is more likely than the latter.

Many thanks to Dick Tahta who commented on earlier versions of this paper.

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SEMIOTICS WORKING GROUP

A report on the meeting at the Institute of Education 9th November 1996
Convenors: Paul Ernest, Exeter University; Adam Vile, South Bank University.

This was the first meeting of this group and was very well attended. It is clear by the level of interest in this group just how fore grounded semiotic issues are becoming within our community. The aim of the group is to bring together those mathematics educators with an interest in semiotics in a context in which ideas, approaches and perspectives can be shared and developed. The spirit of this first session was certainly one of discussion and there was clearly a need to make sense of certain theoretical ideas and notions.

Paul Ernest began the session with a brief theoretical introduction which prompted many questions and much discussion around some difficult ideas. Parallels and differences were drawn between Peircian and Structuralist semiotics and it became clear that in a future session these theoretical perspectives will need further exploration. Reference was made to the work of Vygotsky, much of his theoretical work concerned semiotic notions and specifically semiotic terminology and analysis. Although his work was not discussed in detail at this meeting it will certainly be examined at a future meeting.

A first bibliography of related works was distributed (this may now be found on the www site). Paul then went on to give an example of one way in which a structuralist semiotic approach could be used to assist the analysis of an extract of a student's mathematical writing.

Analysis of text from this perspective is well established in linguistics yet there was much discussion around the question of the formal nature of this methodology. On the one hand it was suggested that there was little benefit to be had from abstracting to more symbolic representations of a discourse yet there was a feeling that the simplification that this abstraction brought assisted in the interpretation of the underlying structure of the text. This issue, the nature and value of formal analysis was flagged as an item for further discussion in future meetings.

An example of an alternative approach to an analysis of text was then presented by Adam Vile. This involved a diachronic look at a number of writing extracts from the history of one student. Initially members of the group were asked to work in groups to attempt an analysis based upon their own reading of the signs and then a partial analysis and a framework for such an analysis was offered by Adam. This form of analysis, in a sense, owns up to subjectivity yet provides a framework for asking questions which may help in "reading the signs".

It was suggested that as an empirical methodology semiotic analysis offers a way of "seeing", discussing and of making new sense of phenomena and that its value may lie in its ability to create useful descriptions. Of course, as is the case with all qualitative methodologies, there are questions of trustworthiness, consistency and credibility of readings of signs and text in any semiotic analysis. The question of how exactly to establish such credibility for semiotic analysis (and how researchers may become better readers of signs) was raised. This is clearly an important issue and one which will be discussed in greater detail at a future meeting.

In summary Paul re-iterated the commitment of the working group to make progress in the clarification and presentation of semiotics in mathematics education. The names and e-mails of the participants were collected and all participants will be contacted before the next meeting for comments and proposals for that meeting.

Future plans of the group:

After the initial meeting, in which the nature of the discussion was general, seSSIOns will be planned with specific foci. For example :

- Semiotic theory
- Semiotics as a research methodology
- Forms of semiotic analysis.
- Possible influences of a semiotic perspective on teaching and learning

Other issues and questions may arise from our survey of participants. We hope to encourage contributions in the form of short presentations, demonstrations and discussion groups.

There is a semiotic working group World Wide Web page at the following address: *http://www.sbu.ac.uk/-vileawa/Semiotic_WG/* .

This will act as the focal point for the group. On this page will be a statement of purpose, description of activities, bibliography, any papers that the group produces, emails and links to other semiotic sites. Of course discussion may also continue via the mathematics education mailing list.

The working group will meet again at the next BSRLM meeting.

British Society for Research into Learning Mathematics

BSRLM is an organisation which acts as a major forum for research in mathematics education in the United Kingdom. It is both an environment for supporting new researchers and a forum for established ones. It is open to and welcomes membership from anyone involved or interested in mathematics education. Membership is open to anyone interested in the area of mathematics education.

BSRLM has an email-list in operation to facilitate effective communication between members and others in mathematics education worldwide. To join this list, send the single word message <subscribe>to<maths-education-request@nottingham.ac.uk>.

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