Children's Conceptions of Number at Ages 10 - 11

Declan O'Reilly
University of Sheffield

This paper gives an account of research aimed at elidting a group of year 6 children's interpretations and representations of number. It raises questions concerning the usual school approaches to number and gives a briif summary of an ahernative 'operational' approach adopted as a result of this study.

Introduction

Reacting to a conference on representation, Belanger (1987) warned of the dangers of neglecting students:

> I have felt during our deliberations that students are strangely absent from our discussion; this is similar to the period of the 1960s when they were absent from the new math and the new science and there is a danger they will be missing from the new representations. One of the things we need to remember is that students construct representations. (1987, p. 105)

In the light of Belanger's warning, this research can be seen as constituting my efforts to ensure that students' contributions were not ignored.

Method

This paper gives an account of a series of interviews aimed at eliciting a group of 10 - 11 year old students' conceptions of number. The students were fIrst interviewed in pairs or individually, in which they were asked questions about natural numbers, rational numbers or directed numbers. They were then asked to represent their numbers in such a way that a younger child would see the differences between their numbers and to record their answers on paper. The representations were collated and reproduced in printed form, and a second set of interviews (in groups of four) was conducted in which they were asked to express their views about the different representations. The interviews, which were audio-recorded, were conducted in the school library . The following account gives a selection of responses from the twenty-six students who were interviewed.

Natural Numbers

By way of a gentle introduction to the interviews, students were simply asked to choose two numbers (Ql), and then to illustrate their choices with drawings. All of the students chose whole numbers, which would seem to suggest that, for them, the concept of number was equated with whole number. Moreover, most of the selected numbers were relatively small (all but one were within the range 1 1 (0), suggesting further that the numbers chosen were those within their experience.

Their responses to the illustration question indicated that the relationship between what was represented (the signifIed) and the mode of representation (the signifIer) was more complex than had previously been supposed. It seemed that for some students, it was suffIcient to depict the 'bigness' of one number relative to the other - they felt no need to accurately preserve the absolute size of each number. Secondly, there was some ambiguity as to what the representations were meant to convey. There was also some doubt regarding whether the visual representation was intended to be a prop for the accompanying symbol or meant to stand independently of it. In the follow-up discussions, the students themselves critiqued the limitations in their own and other's representations.

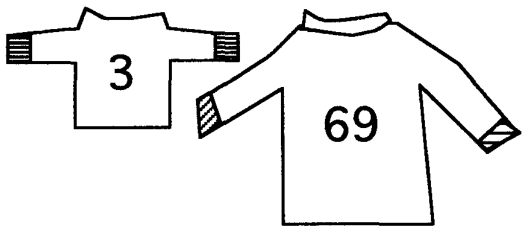

Figure 1a: Hannah's drawing (of jumpers) to compare 3 and 69.

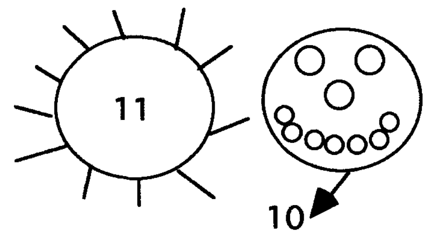

Figure 1b: Ann-Maria's drawing to compare 11 and 10.

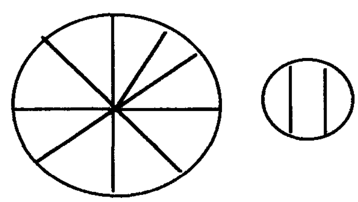

Figure 1c: Blake's drawing to compare 9 and 3.

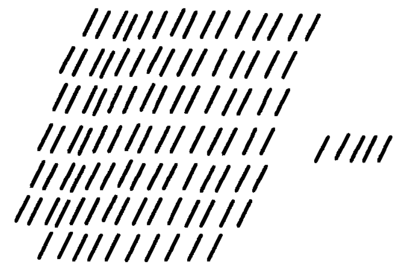

Figure 1d: Dillon's drawing (of rain) to compare 100 and 5.

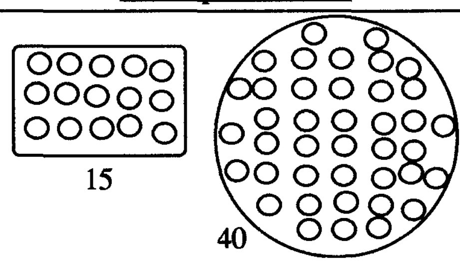

Figure 1e: Tahir's drawing to compare 15 and 40.

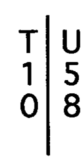

Figure 1f: Joanna's 'drawing' to compare 15 and 8

Figure 1: Students' representations to compare whole numbers

Extract 1:

Speaker	Dialogue	Commentary
Maker:	I like this one. I like that because it shows size.	Points to the drawing of jumpers. [Figure 1a].
Hannah:	I don't think that's a very good idea.	
Interviewer:	You don't think your own idea is very good now. Why not?	
Hannah:	Because if it was a bigger number, this would be like gigantic.	Refers to the difficulties in keeping the drawings in proportion to the numbers.

Hannah, who had drawn the picture of jumpers (Figure Ia), reacted to Maker's preference for it by pointing out what she thought was its weakness. Figure Ia was also criticised by another child because "It is not clear what the numbers stand for." Figure Ib was criticised because "One is for Summer and one is for Winter, but what's that got to do with 10 and 11?" Figure Ic was thought defective because "The parts aren't the same", and with Figure Id, the question of what the representation was intended to signify was raised.

Extract 2:

Speaker	Dialogue	Commentary
Interviewer:	[to Rebecca] Why do you not like it so much?	Refers to Figure 1d
Rebecca:	It would be hard to count.	
Cerisse:	Yes, but even if it is hard to count, they'll still see the difference because of the amount of raindrops.	Refers to depiction of relative sizes.

Liam, who liked the drawing of stones (Figure Ie), was countered by Joanna leading to a discussion on whether a diagram should be able to convey its meaning without the need for back-up writing.

Extract 3:

Speaker	Dialogue	Commentary
Liam:	That one. Fifteen is smaller than forty.	Points to Figure 1e.
Joanna:	Yea, but just looking at that, you couldn't tell the difference between thirty-nine and forty.	
Liam:	But you can write the numbers next to them.	
Joanna:	But, in that case, you can just write them.	

Joanna's implicit use of an abacus mirrors a common teaching approach to place value. Before drawing/writing it, she asked "Do you want a 'drawing-drawing', or can I write it?" This hints at an intermediate interpretation of this representation, i.e., between that of a drawing and the purely symbolic form of a number. Jessie, chose this diagram (Figure If) saying: "If they understand about tens and units, that one would be quite good". This was also the response of Rosie, who said: "If children already knew a lot about numbers, that one would be a good idea. "

Rational Numbers

As part of the protocol, I asked students to select a number between their two whole numbers answers to Question I, and I continued to ask this question until they gave two successive integers. I then asked them to choose a number between these. A frequent first response at this point for many students was to deny the existence of such numbers. Their later answers, which followed researcher interventions, reinforced the contention that the term 'number' was seen - at least initially - as synonymous with whole number. It was very noticeable that all of the students in one group gave fractional answers and most of these were obtained by employing a bisection strategy. It suggested that their notion of fractional quantities was tied to the physical operation of halving. I continued to ask this question until students could give no other non-integer or indicated that they knew the process went on indefinitely. They were then asked to illustrate their choice with drawings.

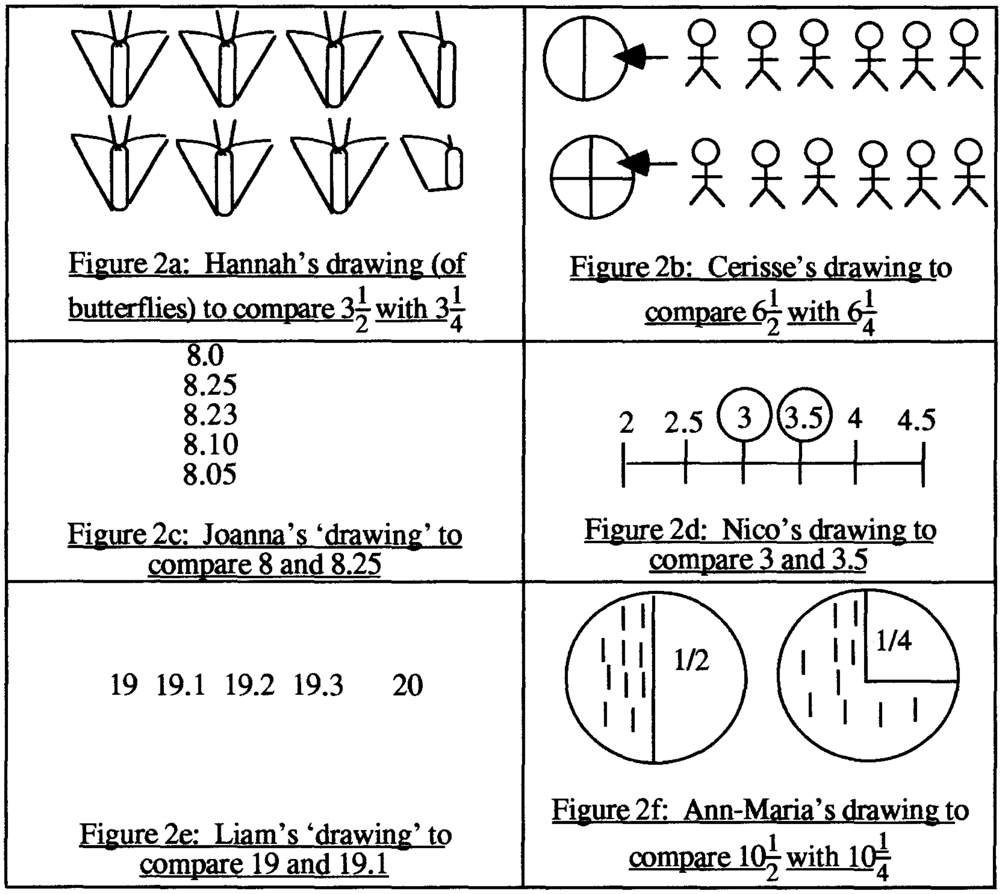

Figure 2a: Hannah's drawing (of butterflies) to compare $3\frac{1}{2}$ with $3\frac{1}{4}$

Figure 2b: Cerisse's drawing to compare $6\frac{1}{2}$ with $6\frac{1}{4}$

Figure 2c: Joanna's 'drawing' to compare 8 and 8.25

Figure 2d: Nico's drawing to compare 3 and 3.5

Figure 2e: Liam's 'drawing' to compare 19 and 19.1

Figure 2f: Ann-Maria's drawing to compare $10\frac{1}{2}$ with $10\frac{1}{4}$

Figure 2: Students' representations to compare rational numbers

Often students chose representations for fractions which were similar to those used to compare whole numbers even when these are clearly inappropriate. An answer which included half a person or half a car seemed to be regarded as quite acceptable. Hannah was the exception to this. She later described her own diagram (Figure 2a) showing portions of butterflies as "Disgusting". Students also mixed representations within the same drawing as in Cerisse's case (Figure 2b), where stick people were used for whole numbers, but fractions were represented by parts of a circle. Another phenomenon observed was for students to represent the fractional parts only, ignoring the whole number part. In general, the students experienced some difficulties in conjuring up a representation for non-whole numbers. This may be due to the novelty of the idea, i.e., representing numbers rather than manipulating them. In the follow-up interviews, criticisms of representations tended to be linked to the accuracy of the representation of the fractional part. Rosie's comment on Figure 2a was typical:

"Because if you were trying to show a small fraction, like a tenth, you wouldn't be able to see it. "

Directed Numbers

Directed numbers are not part of the school curriculum at this age, so questions were framed around temperature. Having chosen two temperatures, the students were asked to make a drawing that would help a younger child to understand the difference between the two.

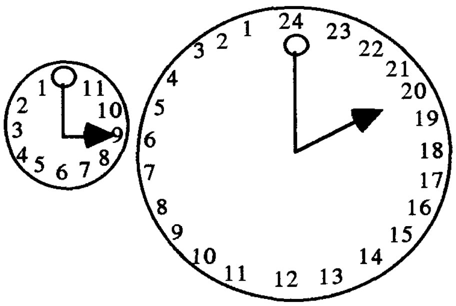

Fig 3a: Ann-Maria's drawing to compare temperatures of 9 and 20 degrees

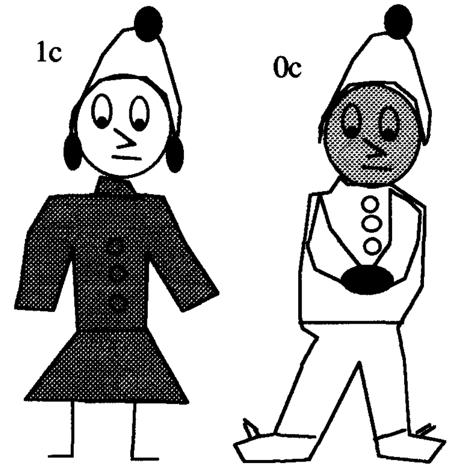

Figure 3b: Courtney's drawing to compare temperatures of 1 and 0 degrees

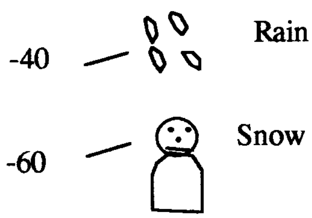

Figure 3c: Jason's drawings to compare temperatures of -40 and -60 degrees.

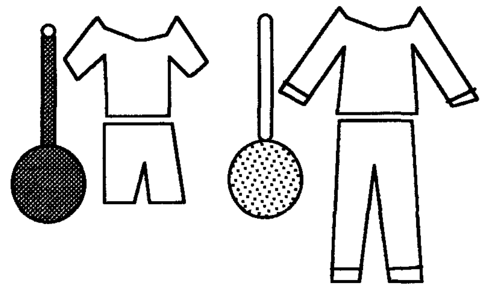

Figure 3d: Hannah's drawings to compare temperatures of 0 and -20 degrees

Figure 3: Students' representations to compare directed numbers

Not surprisingly, many students drew thermometers, but often these were graduated in a way that gave no clear indication of the difference between their two numbers, or not graduated at all. In this sense, they might best be considered as labels for temperature. Hannah's drawings of summer and winter clothes (Figure 3d) alongside thermometers to denote the difference between her two temperatures of 0 and -20 degrees lends weight to this labelling conjecture. The use of light and warm clothing was also used by Courtney (Figure 3b), who added 'cold' and 'warm' shading to her images. Jason's juxta-positioning of numbers next to images of rain and a snowman (Figure 3c) again draws attention to one of the deficits of this form of representation, namely the lack of clarity concerning the nature of the relationship between the number and its accompanying drawing.

Summary and Implications

The findings show that students' representations of numbers tended to be linked to the particular numbers. In some cases, it was not clear whether diagrams were intended to stand alone and in others, it was uncertain what aspect of number was signified by the diagram. Where whole numbers and rational numbers were mixed, it was often the case that students also mixed representations: one aspect of the diagram to denote the whole number and another to denote the rational part.

Representations for directed numbers were generally quite different than those for positive rational numbers. The desirability of having a single representation for all numbers was not appreciated.

The limitations which students identified in their models of numbers alert us to deficits in our own conventional representations. It is common, for example, to use parts of rectangles or circles to depict fractions, but it is questionable whether such a representation could be easily extended to include large numbers, and it is not at all apparent how directed numbers might be represented in this form. The number line representation may appear to overcome some of these difficulties, but it may be necessary to use a device such as 'zooming-in' or 'zooming-out' for small and large numbers respectively.

The models used in elementary school textbooks to depict number vary according to both the type of number and the operation being modelled. Not unnaturally, there is an implicit assumption that if students are learning about number operations, then there already exists a base level of understanding about the numbers themselves. The evidence of this study challenges that assumption. It shows that while students did not naturally use either directed numbers or decimals, contexts could be found which elicited the use of the former. In the latter case, however, similar contexts, very often elicited the use of fractions rather than decimal expressions. A hypothesis was generated to the effect that 'an operational approach' would confer meaning on such quantities, i.e., subtraction would be the *means* by which directed numbers *acquired* meaning, whilst division would provide an introduction to decimal quantities. An epistemological foundation for such an approach can be located in the formal (Peano's Axioms) extension of the natural numbers to the integers, the integers to the rationals and the latter to the real numbers. (Stewart and Tall, 1976).

References

Belanger, M. (1987). Excerpts from the conference. In Janvier, C. (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics.* NJ: Lawrence Erlbaum Associates, 99 -108.

Carpenter, T., Corbitt, M., Kepner, H., Lindquist, M. and Reys, R. (1981). Decimals: Results and Implications from National Assessment. *Arithmetic Teacher*, 28,8,34 - 37.

Foxman, D., Ruddock, G., Joffe, L., Mason, K., Mitchell, P. and Sexton, P. (1985). *A Review of Monitoring in Mathematics* 1978 -1982. London: H.M.S.O.

Gray, E. and Tall, D. (1994). Duality, Ambiguity, and Flexibility: A "Proceptual" View of Simple Arithmetic. *Jounwlfor Research in Mathematics Education*, 24,2, 116 - 140.

Hart, K. (Ed.), (1981). *Children's Understanding of Mathematics* 11-16. London: Murray. Hiebert, 1. and Weame, D. (1992). Links between Teaching and Learning Place Value with Understanding in First Grade. *Journalfor Research in Mathematics Education*, 23, 2, 98 - 122.

Sfard, A. (1991). On the dual nature of Mathematical Conceptions: Reflections on Processes and Objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1 - 36.

Stewart, I. and Tall, D. (1976). The *Foundations of Mathematics.* Oxford: O.U.P.