# Analysing Proceptual Thinking Through Test Items

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*This paper examines the analysis of students' understanding of algebraic expressions. Certain of the test items were written with the aim of allowing students to demonstrate a conceptual understanding of the use of a letter. They were also designed to allow students to show a conceptual understanding of an expression as a whole. These items have been piloted with* 29 *students and their responses are analysed. The extent to which conceptions can be ascertained from test items alone is discussed.*

### Background

The test was written as one aspect of research methodology into the teaching of algebraic expressions using a cognitive conflict teaching approach (see Bell, 1989). The reasons for concentrating on linear expressions are several. The concepts and understandings involved when working with letters in expressions form a large subset of those involved when working with letters in equations. However, equations have an added complication, that of learning and following procedures to solve the equation. This can distract from one of the main aims of the research, to explore students' understanding of letters.

One aspect of students' understanding are the particular strategies that students may employ when working with expressions. Examples of this could be conjoining letters, such as understanding $x + y$ and $xy$ as having the same meaning, or ignoring brackets so that expressions such as $4(n + 5)$ and $4n + 5$ are interpreted as being equivalent. Booth (1984) and Perso (1991) give extensive lists of such errors. The second aspect that the test items address is the conceptual understanding of the use of a letter, from the letter being ignored through to the use of a letter as a variable (see Ktichemann, 1981) .

The other main aim is to use the cognitive conflict methodology to address students' strategies when working with expressions. This methodology can also be used to address students' conceptions of letters. In this methodology, materials are designed to present ideas that have to be assimilated by the students, as they conflict with their existing knowledge. These conflicts are then resolved by using discussion between firstly pairs, then small groups and finally the whole class. The discussion section of the lesson is then followed by consolidation materials (see Bell, 1989 and Perso, 1991).

A subsidiary aim of the research is to analyse students' understanding of the entire expression and to categorise their responses as showing a conception of the expression as a process, a concept or a procept. The notion of procept is given meaning by Gray and Tall (1991) as "the amalgam of process and concept in which process and product are represented by the same symbolism". To illustrate the notions of procedure, concept and procept, consider the ways in which 2x + 4 could be understood.

If a student is able to think of $2x + 4$ in a procedural way, the expression will only have meaning when a value of $x$ is given. In this case the student can then carry out the procedure of calculating the value of the expression.

If a student has a conceptual perspective of $2x + 4$, the expression has meaning as a whole. The student is able, perhaps, to visualise the expression as a graph or understand the expression as something that can be manipulated algebraically.

A student with a proceptual view has an understanding of both procedural and conceptual aspects of the expression. For example, to evaluate $x + 2 + x + 2$, the student might manipulate this expression first to give $2x + 4$ and then carry out a procedure to evaluate the latter expression.

As Gray and Tall explain, a student who can think proceptually can make the switch between an expression representing a process and it representing a product, with this switch probably taking place subconsciously. This subconscious transition can make analysis using a test item quite problematic, as only the outcome is observed. The student will view the procept flexibly, transferring their perspective to whichever is the most appropriate. They will also be ambiguous in that they will not make explicit the perspective that they are using.

The relationship between procedure, concept and procept is not clear cut. Certainly, having a proceptual perspective implies both procedural and conceptual perspectives. However, it seems possible for a student to develop a conceptual perspective without having a complete procedural understanding. This is particularly evident with the use of computers and graphic calculators, as the technology can carry out the procedures and present to the student only a conceptual image. An example of this happening with graph plotting is given in Hunter, Monaghan and Roper (1993).

There has been previous work on studying students' understanding through test items. Ktichemann (1981) reports on the CSMS test that allows an exploration and categorisation of students conceptions of the use of letters. The last of the categories, the use of a letter as a variable, seems to be the most problematic to assess. Ktichemann acknowledges this and gives a working definition that a letter is a variable when there is a second order relationship between two quantities. Here a student goes beyond seeing a relationship as being satisfied by several, but isolated, sets of numbers. An example of this type of relationship would be where the input and output are given for a number machine and the student can generalise this as giving an output of $3n - 2$ when the input is $n$. From the results reported below, it seems that students are able to have a sophisticated understanding of a letter as a variable, but still show an understanding of an expression in a procedural sense. The work of the CSMS team was extended by Booth (1984), in which she examined in greater detail the common strategies that students employ when working with expressions and equations. These include, for example, the closure of expressions such as $x + y$ to give a single result, such as $z$. Perso (1991) undertook a study that subdivided the concepts and strategies given by Ktichemann and Booth, and aimed at remediation of common strategies and errors using a cognitive conflict approach.

## Piloting the Test

The test was piloted in order to develop items that allow students' understanding of expressions to be analysed. The test itself is intended to serve a number of purposes within the research. It is intended to be used as a pre- and post-test that will allow a measure of change to be taken. It is hoped that the test will allow students to demonstrate their understanding of expressions in three particular areas. These are:- the strategies that students use along with the errors that they make; the way in which they view the use of letters and the way in which students view the expression as a whole.

The test has been piloted in two versions, with each version having questions that were intended to be exactly the same, although the diagrams and wording were forced to be slightly different in some instances. One version had closed answers from which the student had to select one or several responses. Some questions gave alternative expressions and the student had to choose those that correctly answered the question. Other closed questions asked the student to give their interpretation of the expression and their understanding of the use of letters, and in these questions they had to choose the response or responses that most nearly agreed with their views.

The second version of the test had questions that allowed for a freedom of response. Where an algebraic expression formed the desired answer, the student was encouraged to write down as many correct algebraic expressions as possible. However, the conditioning that many students experience in mathematics lessons, that there is only one correct answer, seemed to mediate against this.

The two versions of the test were piloted with one group of students, a year 9 top set at a Leeds comprehensive school. The total number of students who sat the test was 29, with 14 taking the open version and 15 taking the closed version. Their responses were analysed in order to ascertain their conceptions regarding the meanings of expressions, letters and recording answers symbolically. The students' responses were followed up by small scale interviews, where unexpected or unclear answers were discussed in order to shed light on their understanding of letters and expressions.

## Proceptual Thinking

As discussed above, one of the purposes of the research is to try to establish whether a student is able to perceive expressions proceptually. The test may be able to provide some information on this proceptual perspective. Since a vital part of a proceptual understanding is flexibility, it seems important that a student is able to view expressions in a range of contexts in either a procedural or conceptual way, whichever is most appropriate to the question.

The following question provides a good example of the difficulty of interpreting students responses to establish whether their thought is conceptual or procedural. The open version of the question is:-

a) What does $2x + 4$ equal when $x = 6$?
b)What does $4 + 2x$ equal when $x = 6$? c)
What does $2(x + 2)$ equal when $x = 6$?
Explain how the expressions are linked.

Student L eXplained that "The expressions are linked because each answer is 16. So each expression has the same answer but is written differently". This indicates a procedural viewpoint, since the answers have been compared. However the second part could easily mean that L knows that the three expressions are equivalent, i.e. has a conceptual perspective, but he does not have the language to express that viewpoint.

Student W seems to have given a clearer indication that she has a proceptual understanding of expressions, as she explains that "They all involve $2x$ and in a) and b), $2x + 4$ or $4 + 2x$. Also all the *x's* are 6' s."

Does Student 1's response, that *"x* is always 6, and it is always *2x"* show a conceptual understanding of the expression? It certainly seems to go beyond the procedural. To state that all the answers are the same seems to clearly show a procedural understanding. An improvement in the wording of the question would be to ask for links between the answers and links between the expressions as two separate items in order to force students to focus on the expressions themselves.

In the closed version, students have the same three parts to the question, but choose their response in the following way:-

Circle all the sentences that most nearly explain how the answers are linked.

The answers aren't linked at all All
the answers are the same
When I try $x = 10$ I get the same answers for each one
The answers will be the same as each other, no matter what number $x$ is

The choice of closed responses that seems to show a procedural approach is option 3. Option 4 seems to be a clearer indication that the student is thinking conceptually. One student selected options 2, 3 and 4 and another selected options 2 and 4.

The following question was designed to directly assess whether a student had a procedural or conceptual perspective of an expression.

Which expression gives the larger result? Please circle the correct expression.

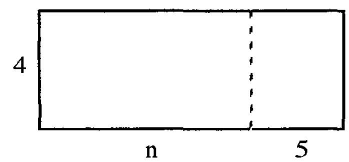
a)$3x + 2$ or $3x + 4$. How do you know that it is bigger?
b)$2x + 4$ or $3x + 4$. How do you know that it is bigger?
c)$3x + 2$ or $2x + 4$. How do you know that it is bigger?

The choice of the word 'bigger' is perhaps unfortunate, as it implies that a numerical value is required to answer the question. Nevertheless, five students answered the first two parts in a proceptual way, for example student L, who said that "$3x + 4$ is bigger because $3x$ is greater than $2x$, but otherwise there is no difference to whatever $x$ equalled $3x$ would be $x$ more than $2x$". This response is typical for part b), in that it shows a proceptual understanding whilst at the same time it ignores the possibility of $x$ being negative. Of these five students, three then went on to answer part c) in a less sophisticated way; by conjoining, by evaluating $x$ and by ignoring $x$.

Student D answered part c) by stating that "It depends upon how big the $x$ number is". This response is sophisticated in terms of Ktichemann' s conceptual categorisation, as she appears to be using $x$ as a variable. The student does not seem to view the expression as a process, but neither does there seem to be evidence that she views the expression conceptually, as she has not compared terms within the two expressions. This illustrates the difficulty of assessing conceptual understanding from a test alone, as the extent of student D's conceptual understanding would have to be refined using an alternative method. Similarly, student M answered the last part by saying "You can't [tell] as $x$ is a variable. If $x$ was 1, $3x + 2 = 5$ and $2x + 4 = 6$, but if $x = 10$, $3x + 2 = 32$ and $2x + 4 = 24$".

The final question is also similar to a CSMS item, although the style of the question follows Booth (1984).

> You can write down the area of this rectangle as $n + 5$ multiplied by 4. Write down as many other expressions as you can for the area of this rectangle.




Student M gave the answer as $5x4 + nx4$, which seems to indicate a process viewpoint. Student D gave the answer of $4(n + 5)$, but is this response qualitatively different to the previous version? Student A gave the response as $4n + 5x4$, which seems more clearly proceptual. The closed version of the question gave several alternative responses, although no student gave the response of $4n + 20$ or $4(n + 5)$ without giving a further response that showed a misunderstanding of brackets.

## Discussion

It is difficult to decide whether a response that seems to go beyond the procedural is in fact conceptual. It would seem evident that a conceptual perspective is gradually developed in the student's mind so that there is a continuum of possible responses from the purely procedural to the fully conceptual. The test items provide only a very blunt instrument in such an analysis.

Similarly, a student who shows a conceptual understanding on a single question may not have a proceptual perspective of expressions as a whole. Proceptual understanding would require a student to be able to use both procedural and conceptual approaches to expressions, depending on what was appropriate in each situation. Of the four questions discussed above, no student showed a conceptual understanding on all four questions and only two students showed a proceptual understanding of three of the questions.

Students' previous experience also make the analysis of responses more difficult. For example, in the last question, many students were unable to correctly use brackets in their answer and so did not show a proceptual understanding of expressions. This may have been because these students had not worked with brackets before, rather than an indication of their conception of an expression.

The results from this small scale pilot indicate a number of areas of difficulty when trying to establish students conceptions from a written test alone. It may be argued that a test is the wrong instrument with which to make these judgments. The rationale for using the test for this purpose is that it is a practical means of collecting and interpreting responses. In order to explore students' conceptions, it seems essential that the results from the test are refined using another approach. Where students show evidence of a possible conceptual understanding, the test could be followed up by selective, small scale interviews that use extension questions to further probe students' understanding of expreSSIOns.

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