Forms and Shadows: formulas and generalisations in A level maths

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The study of 'A' level maths is full of "forms":

- y=mx+c as the equation of a general straight line
- ax^2+bx+c as a general quadratic
- y-Yl =m(x-xl) as the equation of a straight line with given gradient through a given point.

My work with A level students has convinced me that these forms are very important in shaping their thinking and problem-solving strategies. I will offer you accounts of two incidents which have occurred in the course of my teaching in the last year. In each case the validity of what I have to say rests with the reader through an appeal to their experience.

In the first case Robert's treatment of a mathematical problem triggers a recollection of my own recent learning of mathematics and I offer my own experiences as a parallel to his. You are invited to look for parallels in your own experience of teaching and of learning mathematics.

In the second case I offer accounts for the students' reactions to the tasks they are given and teaching gambits which might address the aspects of the students' awareness which I identify. You are again invited to review your own teaching experience for resonance with or reaction against what I have to say.

Robert and y-yo=m(x-xo)

I've asked Robert to work on some questions that I've found interesting. He's already done one. He got quite frustrated with it because he couldn't remember the method he had chosen very well and it didn't seem to be getting him to the answer very quickly. However he was successful finally. Now I give him another problem.

(I give him question 6 *which reads "Find the equation of the tangent to the curve* $y=x^2+1$ *which passes through the origin")*

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R: Okay. Umm .. Oh tangents says derivatives straight away to me - *2x (he writes* ~ = *2x)* umm and so we do *(writes)* $y - y_0$ is equal to .. hmm .. I'm just going up here, it passes through the origin, ...I'm getting all mixed up here ......Well it's going to be, m into $x - x_0$ ....Hmm what's going on here? ....Draw a little diagram here. We're expecting two because the .. graph's in here. We're expecting one there and one here. *(he has drawn a diagram showing the curve and two tangents through the origin)*

L: Umhmm.

R: .. Umm .... *(he has written* $y-b=m(x-a)$*)*

L: What's this point? *(I am referring to the point (a, b))*

R: Umm this is the point which is P here. *(he marks the point of intersection of the curve and the tangent with positive gradient as P)*

L: Right, okay.

(there is a short conversation where Robert tells me that he has tried a question like this before and was not able to do it)

R: ....well if I know that this point .. goes through the origin then surely y_0 and x_0 will be zero ....Oh I don't know ....I'm getting annoyed here.

Robert has no clear idea of how he is going to tackle this question. He is flustered because he has just struggled with another question which I gave him, which he and I thought he would find easy. He recognises that there is something in this question which is like a question he has found very difficult in the recent past.

So he begins by differentiating the function, that is he begins by performing a familiar algorithm which has been triggered by the word "tangent". Next he writes down $y-y_0=m(x-x_0)$, a form for the equation of a straight line which is again very familiar. He spends a lot of time in thought before substituting (a, b) for (x_0, y_0) and subsequently reconsiders whether he should have used $(0, 0)$ instead.

I am reminded of times when I have performed a familiar algorithm in order to see what will happen, or to give myself time to think, or to feel that I am doing something, but with no clear idea of what the outcome of my action will be. I write down a standard form which I hope is relevant to the problem. When I have to decide what will replace the 'standard' parts of this form I realise that its application to this problem is not as transparent as I had hoped. I use a formula which is familiar to me, but I don't feel in control of it. It controls me. I don't know what the outcome will be of the steps I am performing, or what I can do if it does not seem to present another way forward. The objects which I am manipulating are distant and shadowy.

In particular I remember a recent occasion when I was learning some "mathematical methods" and found a problem which required the normal to a surface <1>= 18. I knew that one method of solution involved grad<l> but I wasn't clear how. I calculated grad<l> Then I looked up the solution which told me that grad<l> itself was normal to the surface. I didn't understand why. I can use the formula to produce gradf but I don't know what else I can do with it. I don't feel comfortable with it. I can imagine lots of questions which I cannot answer about it. Its connections with other aspects of my mathematical knowledge are incomplete. My understanding is neither robust nor versatile.

I find the notion of "concept image" (Tall, 1981) useful in describing my understanding of grad. My knowledge of grad is restricted to one type of use and so the potentially rich interconnections with other aspects of my mathematical knowledge have not yet formed. My concept image for grad is very sparse!

Deriving Y-Yl =m(x-xl)

The following is a description from my diary of a lesson from November 1993:

> *Quite early in the lesson I ask them to find the equation of a straight line with gradient* m *and going through the point (Xli* Yl)' *Although we have done two examples of this process using numerical values for* m, Xl *and* Yl *and they have described the process to each other, they find this very difficult. For example, having used the equation* $y_1 = mx_1 + c$ *to find an expression for* c, *Hal substitutes this expression into the equation* Yl=mxl+C *and gets* 0=0. *Tom writes the equation as* y=mx+y-mx. *I ask him whether some of the* xs *and* ys *should have subscripts. He thinks for a while and then adds subscripts to all of them. There is again some difficulty when I later ask them to check their formula with a numerical example. Lloyd begins by substituting the values* 2 *and* 8 *(the given point is* (2,8) *for* x *and* y *rather than* Xl *and* n.

The method which has been developed in this class for finding the equation of a straight line with given gradient through a given point is as follows:

> *First substitute the given gradient for* m *in* y=mx+c. *Next replace* x *and* y *by the co-ordinates of the given point in order to calculate a value for* c. *Then write out the form* y=mx+c *with the appropriate substitutions made for* m *and* c.

There are a number of stories I have used to explain the events I have described here and I have listed them below. Each story is linked with a teacher action designed to enable pupils to work on the awarenesses identified in the story. You are invited to test my explanations against your own experience in any or all of a number of ways. First you may recall students working on similar tasks and decide whether or not my descriptions fit the actions of your students. Secondly you may think about whether the awarenesses I describe are central to the task the students were attempting. Thirdly you may consider whether my explanations of the students' behaviour seem plausible. Fourthly you may, as a thought experiment or in the classroom, tryout the teacher actions I suggest.

Story 1

Throughout this process the form y=mx+c acts as a template into which substitutions are made. Replacement of m and c *by numbers to produce the equation of a particular straight line is familiar and comfortable. But x and* y *may also be replaced by numbers or by other literal symbols. This replacement represents using some information about the line to obtain values for m and/or* c. *Hal had lost sight of the purpose of this substitution.*

Ask students to describe to each other in general the method they are using. Ask about the purpose of each step.

Story 2

In deriving the general form Y-YI =m(x-xI) *students must replace* m *and* c *in* y=mx+c *by literal symbols or expressions involving literal symbols, rather than by numbers. The use of literal symbols blurs the distinction which had existed in working on particular straight lines, between* x *and* y *which in the final equation remained as letters, and* m *and* c *which were replaced by numbers.*

Ask students to anticipate what form their answer is going to take.

Story 3

Having achieved a 'value' for c, *students must decide what to do with it. Looking for somewhere to substitute it back they tend to choose the equation* YI =mxI +c *from which they have just derived their expression, rather than choosing* y=mx+c *itself. In glancing back up the page for some equation to substitute into, their eyes rest first on* YI =mxI +c *because it was written most recently. In fact* y=mx+c *may not have been written down at all.*

Ask students to pause and think about what to do with their expression. Draw their attention to the decision point.

Story 4

The difference between the two equations YI =mxI +c *and* y=mx+c *is insignificant for the students because the* m *and* c *are the salient features of each. In a sense the* x *and yare 'invisible". They are taken for granted because they have always been there. It is* m *and* c *which are noticed because attention in the past has been on replacing them by numerical values. This effect is compounded by the use of* XI *and* YI *as particular values of* x *and* y *because the equations are then superficially similar.*

Compile a suitable list of equations and ask "Which of these is the equation of a straight line?" Ask students to compile their own lists.

Conclusion

I have attempted to highlight some aspects of the learning of sixth form algebra which have struck me in my teaching, without denying the complexity of the learning process for myself or my students. I have used the metaphor of awareness to speak of my students' and my own understanding, and the metaphor of telling a story to describe my sense-making. I have invited the reader to test my descriptions against their own experience in a variety of ways. These stages form the basis of a developing research methodology which aims to capitalise on the individuality and subjectivity of the researcher and the reader. Many of its facets are borrowed from the Discipline of Noticing (Mason 1994, Davis 1992).

My aim has been to allow you to make sense of strands of your own experience by laying them alongside my own accounts. The basis of the validity of my findings is in their capacity to inform the future practice of myself and the reader.

References

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