What is the role of diagrams in communication of mathematical activity?

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In investigating the ways in which teachers assess pupils' texts, eleven experienced teachers were asked to read and assess three piece of student's coursework on tasks set by LEAG (LEAG, 1991). Six of the teachers read work on the 'Inner Triangle' task (investigating the relationship between the dimensions and the area of trapezia drawn on isometric paper) while the rest read work on the 'Topples' task (investigating the point at which piles of Cuisenaire rods starting with a small rod and adding successively larger rods will topple over). All of the children's texts on both tasks contained diagrams of one sort or another. It is the ways in which teachers read these diagrams and the values that they placed on them that I intend to address in this paper.

1 Diagrams as a 'good thing'

The first point to note is that all the teachers at some point expressed approval of diagrams. This approval was several times expressed in general terms, suggesting that diagrams are valued in their own right, regardless of their contribution to the solution or communication of the particular mathematical problem. Charles described his approach to assessment in these terms:

> *Well, shall I say we would do the thing like this, ok. Have a look for children being systematic in their approach, I've got a couple of things written down here .. putting down, tabulating results, perhaps drawing diagrams, putting down ideas, putting forward hypotheses, testing out ideas, urn what else .. perhaps coming up again eventually at the other end with algebraic formulas, things like that, ...*
>
> *(Charles: 4-8)*

Such lists of features are typical of the descriptions teachers gave of their practice. There is little distinction drawn between items related to the act of solving the problem (like hypotheses, testing ideas) and those to do with the form in which the work is presented. According to Charles' self reports of his practice, tables and diagrams, like ideas and hypotheses, do not need any context in order to justify their existence.

While actually reading the children's texts, individual diagrams were, on the whole, ascribed meanings and sometimes values within their contexts; when considering the whole text, however, the mere presence of diagrams in the text appears to be significant in coming to an overall evaluation. Thus, in summing up their

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evaluations of complete pieces of coursework, some teachers summarised by listing an inventory of 'bits', including diagrams, which appear to form part of a list of context free criteria.

> *He's got some patterns here and he's done a few diagrams .. no algebra, oh he's found, yes he has done a formula, he hasn't done an extension.*
>
> *(Fiona: 288-290 Craig)*
>
> *... used a variety of forms ... a results chart. I wonder if it would have been possible to do a graph for it - I don't know without working it through.*
>
> *(Joan: 118-119 Sam)*

Reading as a teacher/adviser (see Morgan 1993), Joan's suggestion that it might have been possible to do a graph seems to be related only to the fulfilment of this criterion in order to gain a higher grade rather than to the quality of the mathematical solution of the problem. One of the few general justifications for this valuing of diagrams was given by Charles:

> *Umm yeah, I mean I think erm there definitely is a need for diagrams. One thing the pupils don't usually draw themselves decent diagrams to help them with problems definitely. But I think given the fact that this is actually a practical thing, you're actually doing that practically, I think there's probably less need for a diagram. Umm, well quite a lot of things you do, I mean, diagrams do count obviously and to actually get them, one of the big problems they've got even with like in the sixth form is that they don't draw diagrams to help themselves work things out. They try and do it all in their head .. they never do .... It's all right but I mean it's not, it's not that valuable. It actually, it doesn't help the problem. No, that's one thing I would look for actually is if they are actually using sensible diagrams to help them with problems and things like that. That's one thing I would look for in assessment.*
>
> *(Charles: 248-259)*

He has some difficulty here in reconciling his knowledge of the specific 'Topples' task as one in which physical objects are manipulated (hence lessening the need for diagrams to help solve the problem) with the general principle that there is "a need for diagrams". This difficulty is manifested in the repeated shifting between the specific and the general. In spite of his admission that, in this case, drawing a diagram "doesn't help the problem", Charles concludes this passage by restating the fact that "That's one thing I would look for in assessment", resolving his difficulties by reaffirming the non-contextualised performance indicator. The absolute value given by these teachers to the drawing of diagrams may have been influenced by their perception that this is something that pupils (even those in the sixth form) do not generally do.

2 Diagrams as signs of 'low ability'

In spite of the value ascribed to diagrams described in the previous section, there were also a number of indications given in the interviews that in some cases diagrams might not be appropriate or might even be a sign of work done by a 'less able' child. Too many diagrams or diagrams which are not arranged according to an easily recognisable system may be taken as evidence that the child was working at a more concrete, practical level. Thus Dan, having described a boy he remembered who had very quickly 'seen' a formula as a solution to the Inner Triangles task, went on to describe in general terms the characteristics of work by other groups of children:

> *But if they don't see that [the formula] then I think 1'd be looking for somebody who would be quite carefully and systematically looking at some diagrams that they could make that had 8 units and then sort of transferring those ideas into 32 triangles. I can't remember .. as this was offered as an Intermediate/Higher level piece of work I don't remember anybody having enormous difficulties. If it was offered lower down, maybe children might get a little bit, might find it a bit difficult, might try varying kinds of diagrams before they actually come up with some answers to that.*
>
> *(Dan: 20-27)*

He has identified a hierarchy of types of children in terms of the way in which they make use of diagrams in attempting to solve the problem. The highest level of abstraction is characterised by an absence of the practical activity which is embodied in the drawing of diagrams. Thus Andy states as one of the things he would look for when assessing that the child should be able to

> *calculate the area of a triangle [sic] just by looking at the dimensions given for any particular trapezium without drawing them and counting.*
>
> *(Andy: 27-29)*

There comes a point in the assessment of a solution to the problem at which the absence of a diagram becomes a positive performance indicator. Similarly, while 'good presentation' is valued, it is also possible that a teacher may interpret some forms of presentation as detracting from the mathematical content of the work. Thus Harry, reading Sally's 'Topples' text, begins by praising her presentation:

> *Three dimension illustrations to start with, very nice, nicely presented.*
>
> *(Harry: 190 Sally)*

but quickly moved on to criticise the form of the presentation:

> *Oh, she's colour coded anyway. She's used some sort of key. Um, not really necessary. She could have just put the numbers on, ...*
>
> *(Harry:* 195-196 *Sally)*

There is a suggestion here that the use of colour is an unnecessary elaboration. Using numbers to label her diagrams would not only have saved her time but might also be interpreted as a more abstract, and hence more highly valued, way of thinking about the problem.

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These qualifications of teachers' reactions to the presence of diagrams and elaborate presentation in children's texts imply that the context free approval of diagrams described above does not reflect the way in which the teachers actually read. Their advice to their pupils to "draw diagrams" may not be helpful to pupils as it does not provide any means of distinguishing between those uses of diagrams which will be approved and those which will be taken as signs of a low level of mathematical achievement or will be judged to have been a waste of time.

3 Some examples of readings of specific diagrams

I shall now move on from general issues to consider teachers' readings of some specific diagrams, attempting to make some connections between the forms of the diagrams and the understandings that the teachers appear to have constructed from them and the values ascribed to them. It is interesting to note that there were in several cases contrasting readings made by different teachers.

3.1 An analytic diagram as evidence of thinking

Richard's Inner Triangles text contains a hexagon with a diagonal drawn in, dividing it into two trapezia (see Figure 1).

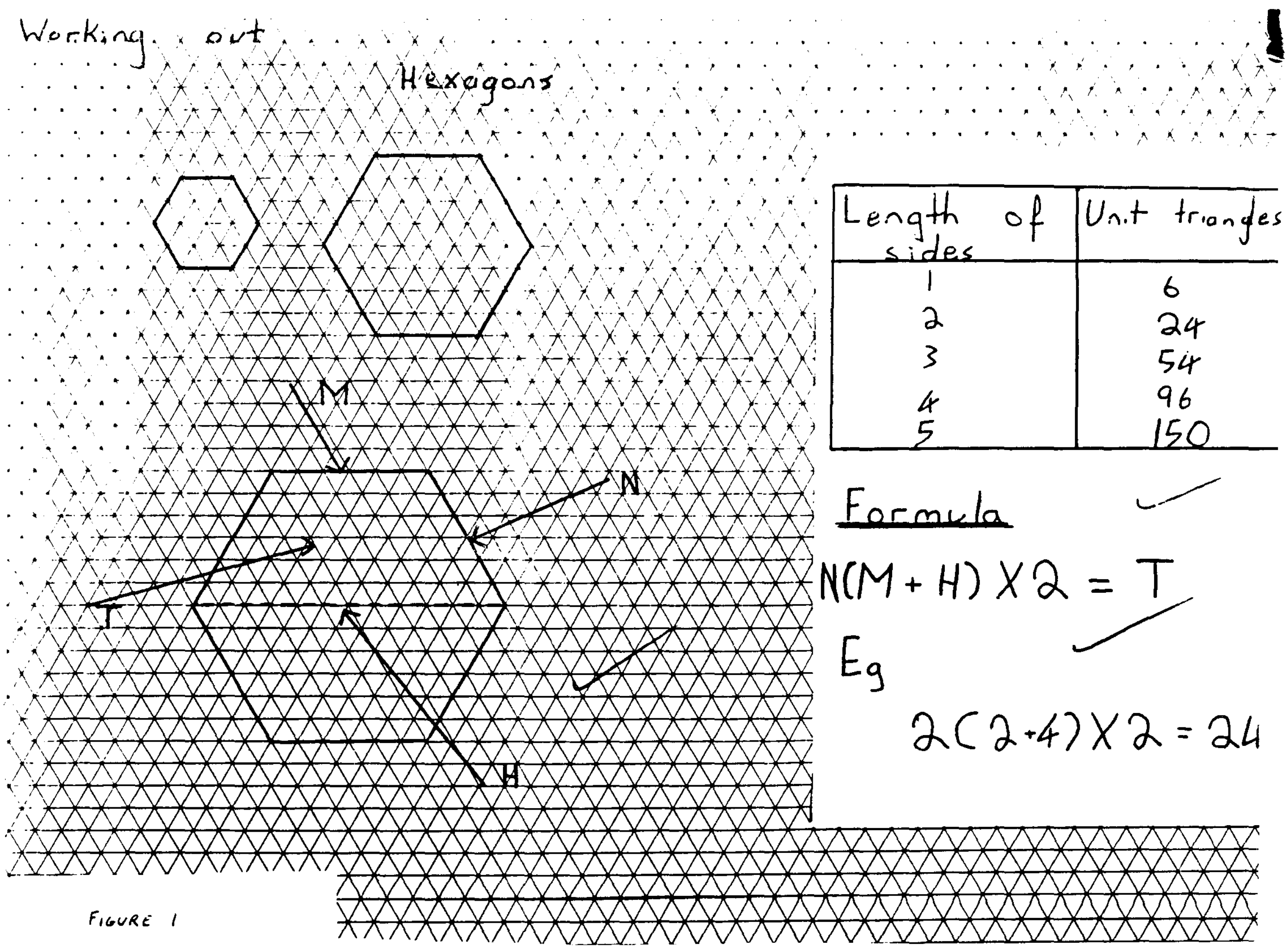


FIGURE 1

The structure of this diagram is analytic and, having no numeric labelling, appears to represent a 'general' hexagon. It may thus be interpreted as making a statement like , A hexagon consists of two congruent trapezia'. This led Joan to comment:

> *I wonder .. the fact that he's drawn that dotted line across the middle makes me think he was looking at it in terms of two trapezia but he hasn't said that here, yet anyway unless it's further on,* so *that seems like a very sensible idea .... , but it would have been a good idea perhaps if he's written in there*
>
> *(Joan:* 143-147 *Richard)*

While the diagram is enough to make Joan think she knows how Richard was approaching the problem, she is not satisfied that the diagram can stand by itself. Her suggestion that "it would have been a good idea" to provide the same information in words may indicate her own uncertainty about the inference she has made but the impersonal construction suggests that her criticism is based on a more general criterion about what she considers to be appropriate methods for communicating such ideas. Similarly, Dan commented on the same diagram:

D *he's moved into hexagons which he's seen as being two trapeziums and therefore fairly simply got*

I *How do you know he's seen them as two trapeziums?*

D *Cos he's drawn it on there. He's taken basically the same formula and multiplied by two, and my guess is that* [..] *that's how he saw it. But again there's no writing whatsoever.*

(Dan: 157-162 *Richard)*

Here Dan makes a connection between the form of the diagram and the formula on the page next to it; this corroboration surely contributes to his greater confidence in the inference he has made. Nevertheless, like Joan, he remarks on the absence of "writing". In this case, given Dan's confidence in his "guess", the criticism of the lack of writing appears to be being made from the position of an examiner concerned with fulfilment of criteria rather than with understanding the student's thinking.
In contrast to Dan's use of Richard's diagram to deduce the way in which the hexagon formula was derived, Andy completely ignores the diagrams on this page:

> *Now .. he's probably got enough evidence there to sh .. to produce that formula. He's produced what appear to be five pieces of data and .. because of his experience with the trapezia in the first part I would be confident that he could produce that formula for the second part. Although it's not apparent just from looking again, it look as if the formula has been produced ...*
>
> *(Andy:* 211-216 *Richard)*

The five pieces of data he refers to must be the entries in the table (as there are only three diagrams altogether) and it is assumed that the formula has been achieved by pattern spotting using the table. Much of Richard's text consists of a series of pages with very similar structures (diagrams, table, formula) for each of the shapes he has considered. This structure may be influencing the way in which Andy is reading each individual page. From Andy's reference to the earlier "experience" on trapezia it appears that he has constructed a picture of Richard's text as a series of selfcontained pieces of work, each of which independently follows the same pattern of working. Any changes in Richard's achievement in later parts of the work may be explained by a greater facility gained through repeating and practising the same set of techniques rather than by the use of earlier work to derive later results in a new way.

3.2 Dynamic counting diagrams

Craig's response to the first part of the 'Inner Triangles' task is presented using roughly drawn diagrams on isometric paper, cut out and pasted onto lined paper so that they are embedded within the written answers to the questions (see Figure 2).

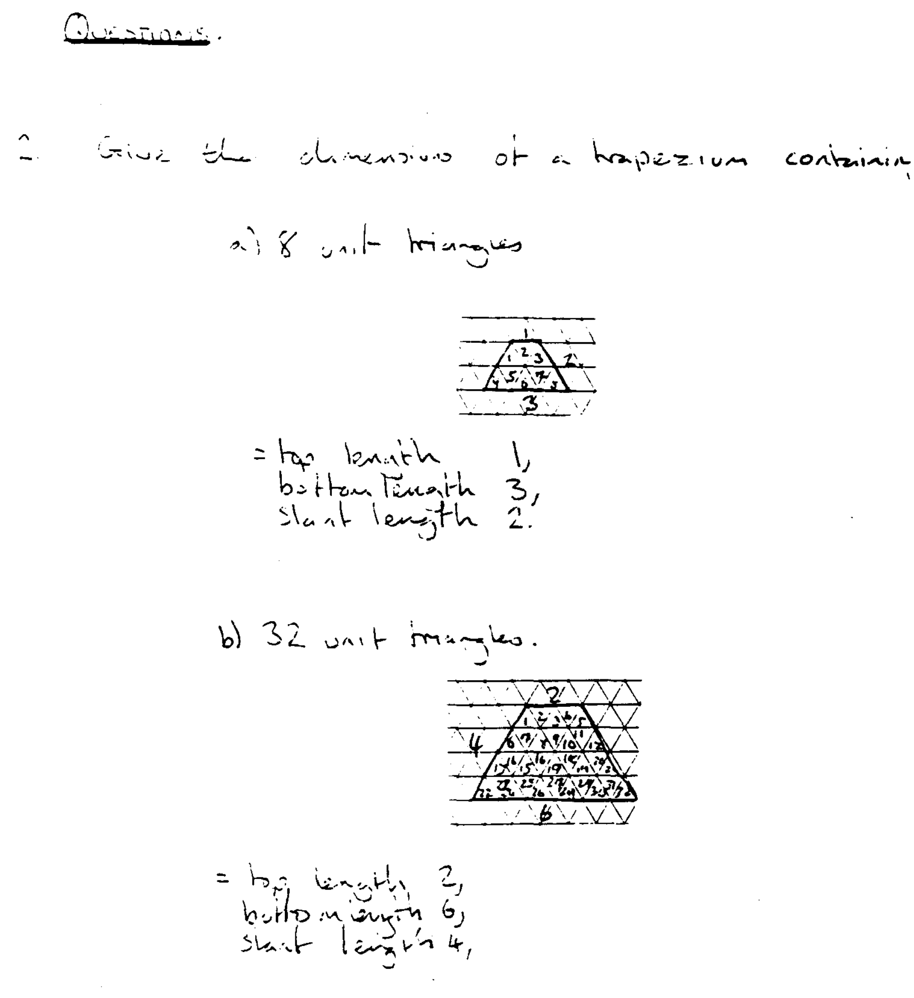


FIGURE 2

The small triangles in each of the trapezia are marked in sequence with numbers, indicating a systematic counting process. This is taken as evidence that this is the way that Craig himself has found the number of triangles in each of the trapezia:

> *Um he's immediately got into that and shown that he is actually working it out and then he's showing clearly that he's got an* 8 *unit triangle and a* 32 *unit triangle it can be no, even without writing, there's no doubt in your mind that he's done it, he's worked it out he's got the answer ... Now again, he's used the same technique in investigating it. He's shown that he's gone from one trapezium to another and he's counted and it's not the same trapezium that he's used, he's used a separate one, although it's a bit small he's actually shown what's happened.*
>
> *(Dan: 263-270 Craig)*

The numbering, together with the 'roughness' of the diagrams suggests for Dan not only that Craig has counted in a particular way, but also that he has done the work himself rather than copying answers. While several other teachers also read Craig's diagrams in a similar way, Joan appears to see them as a method of communication with the reader rather than as a tool for solving the problem:

> *He's started off in quite a nice way with the diagrams next to the working out which is useful.*
>
> *(Joan:* 166-167 *Craig)*

While Dan has read Craig's diagrams as self-contained messages, apparently ignoring the verbal and symbolic part of the text or seeing it as merely a label for the diagram, Joan is integrating her reading of the whole page. The positioning of the diagrams off-set to the right of the questions and answers may suggest that they are re-presentations of that information rather than working out that might have preceded the answers.

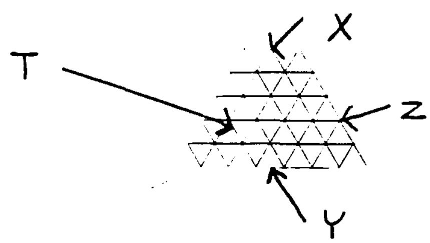
3.3 A **dynamic diagram** as 'explanation'

Richard's diagram of a trapezium has a number of features which mark it as a form of explanation rather than as a piece of data (see Figure 3).

Results

Top length	Bottom length	Slant length	Unit triangles
1	3	2	8
2	4	2	12
3	5	2	16
1	4	3	15
2	5	3	21
3	6	3	27
1	5	4	20
2	6	4	32
3	7	4	40
1	6	5	30
2	7	5	45
3	8	5	55

Formula



$$Z(X+Y) = T$$

FIGURE 3

Firstly, its position on the page after the table of results for the number of inner triangles in a trapezium means that, even if the page is read as a narrative of the author's method of solution, the diagram cannot be taken as chronologically prior to the table. It is also labelled with variable names rather than with numbers; it must, therefore, be read as a generic trapezium rather than a specific example. The arrows which connect the variable names to the corresponding parts of the diagram stress the importance of these labels; they may be read as instructions to the reader to pay attention to the way in which the diagram is labelled. This interpretation appears to coincide with the readings by Andy:

> *He's indicated on a diagram what the variables stand for and that's fine*
>
> *(Andy:* 187-8 *Richard)*

and by Joan, although she is initially less specific about the diagram's role:

> *And he's explained it quite nicely with the diagram.*
>
> *(Joan:* 136-7 *Richard)*

In contrast, Fiona, looking for evidence of working out, did not even recognise the existence of the diagram:

> *This is a major problem because he's got these results but unless one is there in the class and you're a teacher you don't know whether this is his results or somebody else's. He hasn't shown any diagrams of where these results have come from. He hasn't done any drawings as far as* I *can see. He's come up with a formula which is* z *equals.* Z *must be the slant height.* Is *equal to x plus y equals* t.
>
> *(Fiona:* 57-61 *Richard)*

Fiona is looking for diagrams as evidence of data generation. Because this diagram does not fulfil Fiona's expectations about the functions of diagrams within the genre, its existence is not even acknowledged. In spite of the arrow joining the variable name Z to the slant height of the diagram, she appears to have to construct this correspondence for herself using her existing knowledge of the problem rather than reading it from the diagram.

Dan, on the other hand, accepted this diagram as evidence of working out by constructing a story about how Richard might have made use of the single diagram to generate several pieces of data:

> *He's written out a table and although there's not a great deal of, there doesn't appear to be a great number of diagrams he could have used the diagrams as he was going along, in other words he could have blanked off some of them to get his .. to get his table.* [demonstrates how to use a single diagram to represent many trapezia by covering part with his hand] *It's a little bit difficult to see where he got perhaps all of these but he might well have done them from as he was working along the shape. He's trying a systematic system of working by doing the one two three one two three making the top length, looking at the top length, extending the top length and that's why I'm suggesting that you don't necessarily have to draw simple diagrams all the time. Children* <u>*do*</u> *but it wasn't actually necessary in this piece of work.*
>
> *(Dan: 105-115 Richard)*

In this case, the very lack of diagrams has led Dan to hypothesise about the method Richard might have used. He had previously identified Richard as of 'high ability', which may have influenced the way in which he has interpreted this minimalist diagram.

The multiplicity of readings of this single diagram is indicative of the variety of expectations that teachers bring to their reading and of the influence of these pre-existing expectations on the meanings that they construct from the text.

3.4 Naturalistic diagrams as signs of practical activity

In her response to the 'Topples' task, Sally drew a large number of diagrams showing three-dimensional representations of the piles of rods (see Figure 4).

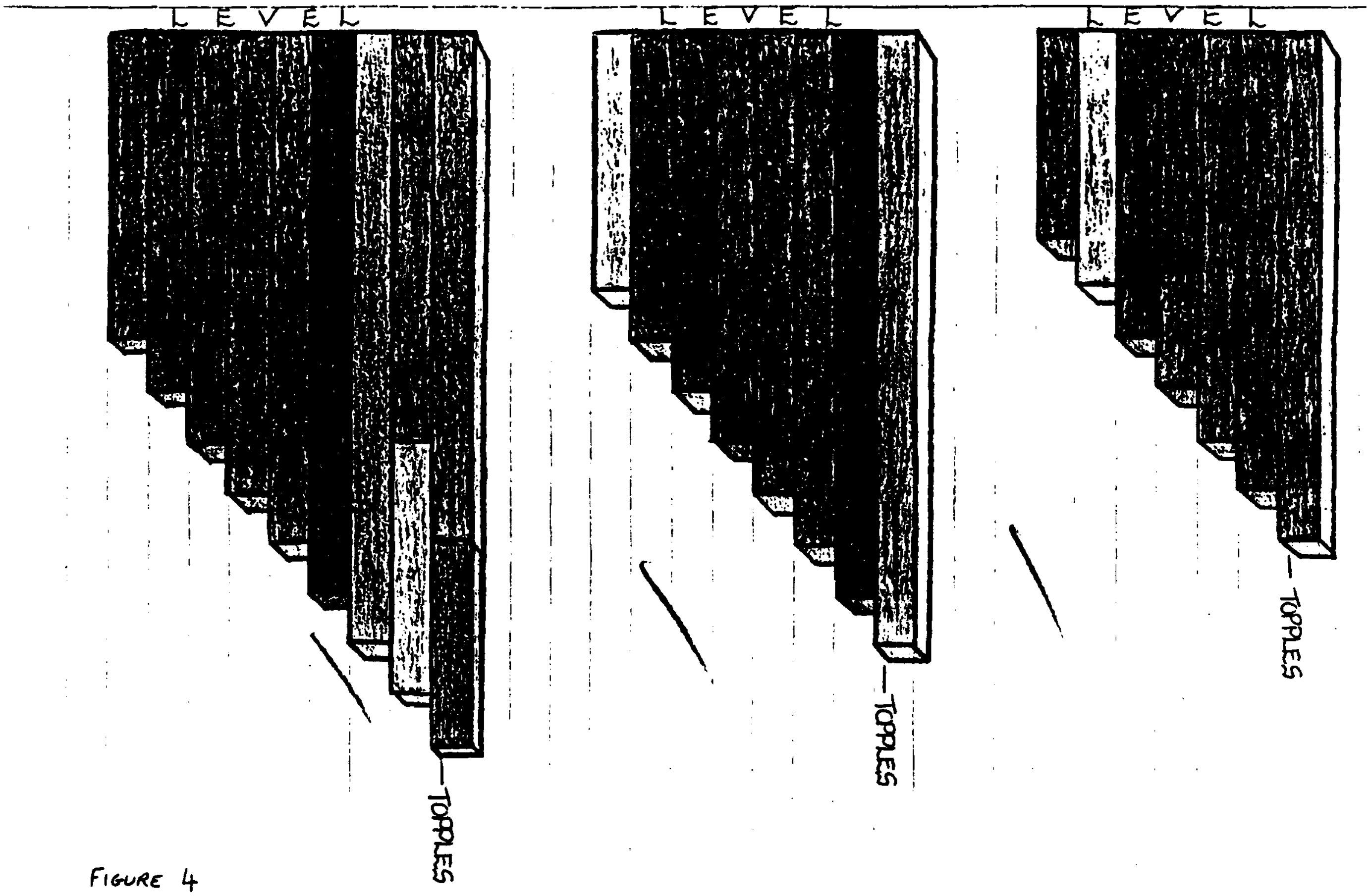


FIGURE 4

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The naturalism of these diagrams is enhanced in two ways: firstly each rod is co loured (using the same colour scheme as the original Cuisenaire rods used for the practical activity) rather than labelled with numbers; secondly, where the length of a rod in a pile is greater than 12 units it is drawn as a combination of two smaller rods. This again corresponds to the structure of the concrete materials used by all the pupils whose work was read and by the teachers in preparing for their interviews. Although the other pieces of work on this task also contained the results for piles that must have been built in this way, neither included diagrams for any but the smallest piles.

As shown earlier, Sally's presentation was appreciated aesthetically by the teacher-readers but was simultaneously devalued as over elaborate and a waste of time. The naturalism of her diagrams, however, seems also to have contributed to the way in which Harry made sense of her mathematical activity. Whereas in the case of the other two texts, the activity was presented and read as being largely about abstract patterns of numbers, Harry questions the validity of Sally's results based on the story that he has constructed about her practical activity.

H *Now one of her illustrations here shows I think that she's used, she's actually using split rods. We haven't actually tried that but I'm sure that would have an influence on the result.*

I *Probably the rods only go up as far as twelve don't they*

H *Yeah [laughs]* I *wonder what she makes of that. One that - I'd be interested to see if the results show anything different.*

(Harry: 198-203 Sally)

Although the constraints of the materials used mean that the other pupils and Harry himself must also have used split rods, the effects of this on the results generated by the practical activity are only considered in Sally's case. The naturalistic form of her diagram creates a story about the nature of her activity:

> *[Diagrams] ... represent events taking place over time as <u>spatial configurations</u> and* so *turn 'process' into 'system' - or into something ambiguously in between . . . . Diagrams are therefore akin to expository writing, while naturalistic images ... are akin to story writing.*
>
> *(Kress & Van Leeuwen 1993)*

This story is used to explain why her results are different from those produced by the others:

I *Yeah but she actually came up with different answers there as well in her original data*

H *Mmm ....* I *think this is just going to go back to what was said about the blocks being split.* So I *think that's had an effect there. Um .. when it's five at the bottom .. yeah she's got twelve um I'm not convinced that would make it topple. But in her illustration, is that is that I can't see which one.*

I *This is the one with five on the bottom. That isn't split.*

H *Right.* So *I'm not sure that that's - I'm not sure that it would topple then. But if she's convinced that it did topple - again we said about the accuracy of the actual modelling of it the setting up of it. If it toppled it may be something to do with it but she's convinced that it toppled. Then we have to accept that. Um but it's the six one the next one I think where the problem may be. Because she feels that she's gone, she's extended it to one next level and she's spotted that it's still two extra um that I think that's what's made her convinced that that is the correct solution. Um, but it would be this, and I would I would - post investigation again I would try to see if I can get any - go down to CDT or whatever and get some blocks of wood and chop them up and see you know if it does make a difference. I would get actually longer lengths um to try and eradicate this.* So *that's what's done it* I *think. That's what's caused the damage.*

(Harry: 213-229 *Sally)*

Although he suggests alternative sources for Sally's errors - general inaccuracy in building the piles even when they are not split or blindly continuing a pattern of numbers - the remedy he prescribes for her difficulties is still based on the persistent idea that the split rods are causing the errors. The naturalistic form of Sally's diagrams so strongly presents her work as being of a very concrete nature that this overrides all the other possible interpretations that Harry had identified.

6 Summary

When considered out of context, diagrams are seen by teachers to be valuable in their own right. This value seems to be ascribed to them either because they are seen as a demonstration that a pupil has the ability to use a variety of forms of mathematical communication or because they are believed to be helpful in solving problems. Within the context of a particular problem, however, the use of a diagram to help solve the problem may be given a low value by being seen as a sign of a concrete, practical approach rather than a more prestigious abstract approach to the problem. Naturalistic diagrams in particular may give rise to a reading of the author's activity as concrete; such a reading may affect the ways in which the rest of such a text is interpreted.

From the variations in the interpretations of the same diagram by different teachers, it is clear that diagrams are not 'transparent' unambiguous forms of communication. Readers' expectations about the nature of the text that they are reading and about generically appropriate functions of the diagrams within that text are influential in shaping the meanings that they construct from a diagram and the values that they place upon it. Indeed, if the form of a diagram diverges too far from the reader's expectations, it may be ignored or passed over with little effort to incorporate it into a coherent reading of the whole text.

The variability in the teachers' readings of diagrams in the children's texts calls into question the general advice to draw diagrams that many of them give to their pupils. While the mere presence of a diagram in the text may be taken as a positive performance indicator the meanings ascribed to that diagram seem to be more important in contributing to the teacher's evaluation of the text as a whole and of its author. Where the diagram does not conform to the teacher's expectations of the genre it may be passed over without making a significant contribution to the meanings constructed by the reader; where it is taken to indicate a concrete way of working it may lower the teacher's judgement of the whole text. The teachers' general approval of diagrams when expressed out of context shows little conscious account being taken of the variability of the ways in which diagrams may be used or of the variability in their own judgements about specific diagrams read in context. In this general form, it cannot therefore be useful to pupils. Indeed, by following the advice to draw a diagram, it is actually possible that a pupil may lower the teacher's evaluation of their achievement. In order to be useful, such advice needs to form part of a more general awareness of the features of the coursework genre as a whole and of the functions that diagrams may play in teacher's constructions of meanings within this genre.

References

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